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SCALED BOUNDARY FINITE ELEMENT FORMULATIONS FOR LAMINATED PLATES

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ABSTRACT

In this paper, a new scaled boundary finite element for arbitrary laminated plates is presented. The scaled boundary finite element method [1] is a finite element formulation employing a discrete form of the KANTOROVICH reduction approach formulated in scaled boundary coordinates. If the system of ordinary differential equations for the unknown functions is solved in a closed-form analytical manner, this efficient approach yields a field expansion in scaling direction [2, 3]. Although the method has been applied successfully to many problems of continuum mechanics, its application to plate bending problems is, apart from an initial approach to the dynamics of thin isotropic plates [4], rather unexploited. This study provides a new extension of the method to the static analysis of arbitrary laminated plates.

Utilising first order shear deformation theory, the virtual work balance for a laminated plate domain Ω with the boundary Γ in the absence of volume loads is given by the equation

$$\begin{aligned}\int_\Omega \Big[& (\boldsymbol{\mathcal{L}}\delta\mathbf{u})^T \mathbf{D}_A \boldsymbol{\mathcal{L}}\mathbf{u} + (\boldsymbol{\mathcal{L}}\delta\mathbf{u})^T \mathbf{D}_B \boldsymbol{\mathcal{L}}\boldsymbol{\phi} + (\boldsymbol{\mathcal{L}}\delta\boldsymbol{\phi})^T \mathbf{D}_B \boldsymbol{\mathcal{L}}\mathbf{u} + (\boldsymbol{\mathcal{L}}\delta\boldsymbol{\phi})^T \mathbf{D}_D \boldsymbol{\mathcal{L}}\boldsymbol{\phi} \\ & + (\boldsymbol{\nabla}\delta w + \delta\boldsymbol{\phi})^T \mathbf{D}_S (\boldsymbol{\nabla} w + \boldsymbol{\phi}) \Big] \mathrm{d}\Omega \\ = & \int_{\Gamma_1} \delta u_n \bar{N}_n \mathrm{d}\Gamma + \int_{\Gamma_2} \delta u_t \bar{N}_t \mathrm{d}\Gamma + \int_{\Gamma_3} \delta\phi_n \bar{M}_n \mathrm{d}\Gamma + \int_{\Gamma_4} \delta\phi_t \bar{M}_{nt} \mathrm{d}\Gamma + \int_{\Gamma_5} \delta w \bar{S}_n \mathrm{d}\Gamma \ , \end{aligned} \tag{1}$$

with in-plane displacements $\mathbf{u}$, out-of-plane displacements w and rotations $\boldsymbol{\phi}$ as kinematical variables and differential operators $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\nabla}$. The material matrices are denoted by $\mathbf{D}$ while the symbols with overbars denote boundary traction resultants in normal and tangential directions. For thin plates with suppressed shear deformations, the KIRCHHOFF kinematical assumption ($\boldsymbol{\nabla} w + \boldsymbol{\phi} = \mathbf{0}$) holds.

A scaled boundary coordinate system (see Figure 1) with dimensionless ξ-η-coordinates ($\xi_i \leq \xi \leq \xi_e$, $0 \leq \eta \leq 1$) is introduced and the domain is mapped to the discretised boundary ($\xi = 1$)

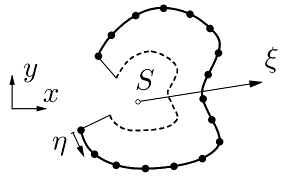

Figure 1: Scaled boundary discretisation.

using the transformation

$$\mathbf{x} = \mathbf{x}_0 + \xi \hat{\mathbf{N}}(\eta)\mathbf{x}_s \quad , \tag{2}$$

with similarity centre $S(x_0, y_0)$ and geometry shape functions $\hat{\mathbf{N}}(\eta)$. If a mapping of the total domain is not possible, appropriate subdomains need to be introduced. For the differential operators, an additive decomposition of the form

$$\boldsymbol{\mathcal{L}} = \hat{\mathbf{B}}_1(\eta)\frac{\partial}{\partial \xi} + \frac{1}{\xi}\hat{\mathbf{B}}_2(\eta)\frac{\partial}{\partial \eta} \quad , \quad \boldsymbol{\nabla} = \hat{\mathbf{B}}_3(\eta)\frac{\partial}{\partial \xi} + \frac{1}{\xi}\hat{\mathbf{B}}_4(\eta)\frac{\partial}{\partial \eta} \quad , \tag{3}$$

is used, where the matrices $\hat{\mathbf{B}}(\eta)$ contain components of the inverse Jacobian matrix evaluated at the discretised boundary. The displacement fields are approximated in separable product form by

$$\mathbf{u}(\xi, \eta) = \mathbf{N}_u(\eta)\mathbf{u}_h(\xi) \quad , \qquad w(\xi, \eta) = \mathbf{N}_w(\eta)\mathbf{w}_h(\xi) \quad , \tag{4}$$

with displacement shape functions $\mathbf{N}(\eta)$ and unknown functions $\mathbf{u}_h(\xi)$ and $\mathbf{w}_h(\xi)$.

In order to derive the governing equations for a laminated thin plate element, the virtual work equation (1) is transformed to the scaled boundary discretisation using the relations derived above. The factors depending on η yield the boundary stiffness matrices by integration along the boundary. Integrating by parts twice yields a system of ordinary differential equations

$$\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{E}}_{A0} & \tilde{\mathbf{E}}_{A1} & \tilde{\mathbf{E}}_{A2} \\ \tilde{\mathbf{E}}_{B4} & \tilde{\mathbf{E}}_{B5} & \tilde{\mathbf{E}}_{B6} & \tilde{\mathbf{E}}_{B7} \end{bmatrix} \begin{bmatrix} \xi^4 \mathbf{u}_{h,\xi\xi\xi} \\ \xi^3 \mathbf{u}_{h,\xi\xi} \\ \xi^2 \mathbf{u}_{h,\xi} \\ \xi \mathbf{u}_h \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{E}}_{B0} & \tilde{\mathbf{E}}_{B1} & \tilde{\mathbf{E}}_{B2} & \tilde{\mathbf{E}}_{B3} \\ \tilde{\mathbf{E}}_{D0} & \tilde{\mathbf{E}}_{D1} & \tilde{\mathbf{E}}_{D2} & \tilde{\mathbf{E}}_{D3} & \tilde{\mathbf{E}}_{D4} \end{bmatrix} \begin{bmatrix} \xi^4 \mathbf{w}_{h,\xi\xi\xi\xi} \\ \xi^3 \mathbf{w}_{h,\xi\xi\xi} \\ \xi^2 \mathbf{w}_{h,\xi\xi} \\ \xi \mathbf{w}_{h,\xi} \\ \mathbf{w}_h \end{bmatrix} = \mathbf{0} \quad , \tag{5}$$

and a set of algebraic equations for the dynamic boundary conditions. The constant matrices $\tilde{\mathbf{E}}$ depend on the boundary stiffness matrices only. This homogeneous Euler-type system has a general solution of the form

$$\mathbf{u}_h = c_1 \lambda_1 \xi^{\lambda_1 - 1} \boldsymbol{\psi}_{u1} + c_2 \lambda_2 \xi^{\lambda_2 - 1} \boldsymbol{\psi}_{u2} + \cdots + c_n \lambda_n \xi^{\lambda_n - 1} \boldsymbol{\psi}_{un} \quad , \tag{6}$$

$$\mathbf{w}_h = c_1 \xi^{\lambda_1} \boldsymbol{\psi}_{w1} + c_2 \xi^{\lambda_2} \boldsymbol{\psi}_{w2} + \cdots + c_n \xi^{\lambda_n} \boldsymbol{\psi}_{wn} \quad , \tag{7}$$

including power-logarithmic terms and terms due to conjugate-complex pairs of exponents. The modal vectors $\boldsymbol{\psi}$ and the exponents λ are determined by solution of the corresponding eigenvalue problem. At the boundary, the nodal kinematic degrees of freedom are enforced to calculate the constants c_i. The stiffness matrix is obtained by introduction of this special solution into the algebraic equations for the dynamic boundary conditions. Using appropriate solution subsets, bounded as well as unbounded domains can be treated. As will be demonstrated by numerical examples, this new and efficient scaled boundary element for laminated plates is especially well suited for stress concentration problems.

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