A spectral volume Navier-Stokes solver on unstructured tetrahedral grids

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ABSTRACT

The spectral volume (SV) method was first introduced as a method to solve systems of convection equations, like the Euler equations, in a series of papers by Wang et al., e.g. [1] and the references therein. It is related to the discontinuous Galerkin (DG) method, in the sense that it also uses high-order polynomials to approximate the solution in each grid cell, and Riemann solvers to deal with the discontinuities in the solution along the cell faces. The SV method can be extended to systems of convection-diffusion equations, like the Navier-Stokes (N-S) equations, in a similar manner as the DG method, see [2]. Further contributions to the development of the SV method were made in Van den Abeele et al. [3,4], where the method stability was analyzed and stable schemes were derived for 1D and for 2D triangular grids. In the present contribution, an implementation in the COOLFluiD code [5], which was developed at the Von Karman Institute, of the SV method for the N-S equations on tetrahedral grids is presented. The implementation features both the 'quadrature' approach, where Gaussian quadrature formulas are used for the evaluation of the residual integrals, and the 'quadrature-free' approach, where such formulas are avoided. The latter approach was recently described for the 2D Euler equations in Harris et al. [6]. In the present implementation, for an order of acccuracy higher than two, a significant decrease in computational effort for the evaluation of the residuals was observed with the 'quadrature-free' approach, while the order of accuracy was maintained. For the discretization of the diffusive terms, an approach similar to the 'local approach' for the DG method, as described in Cockburn and Shu [7], was followed. As an example, the mach contours for the flow around a NACA0012 airfoil at $Re = 5000$ and $M = 0.5$, obtained with a third-order SV scheme, is shown in the left plot of Figure 1.
 While high-order accurate compact schemes, such as the SV and the DG methods, can yield accurate results more quickly than traditional low-order schemes, fast and robust solvers are a necessity to fullfill this potential. This is illustrated in Figure 2, where the Fourier footprints (FFs) corresponding to second-, third- and fourth-order accurate SV schemes for the 1D linear diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2$ are plotted. These FFs were computed in an analogous way as described in [3]. It is obvious from the plots that the size of the FF increases dramatically with the polynomial order of the SV schemes. Consequently, there is a severe restriction on the maximum time step that preserves stability, if traditional explicit Runge-Kutta (R-K) schemes, as described in [3], are used as solvers for these schemes. Therefore, an implicit solver based on the backward Euler scheme was implemented. This scheme leads to a
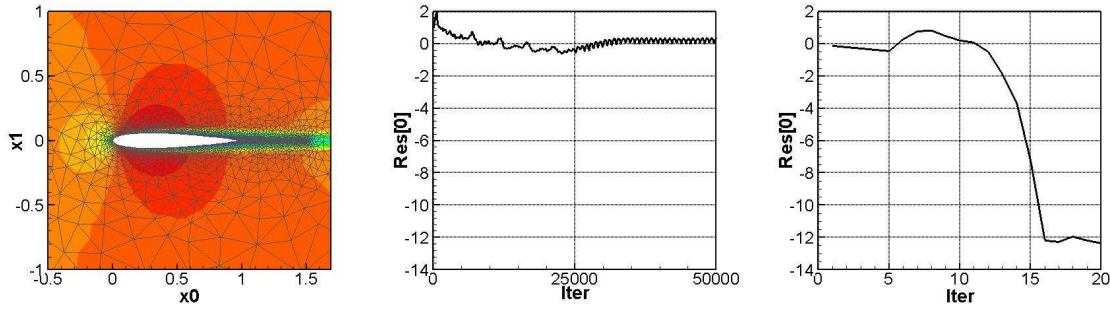
Figure 1: Mach contours ($\Delta M = 0.025$) (left), and convergence history with an explicit R-K solver (middle) and an implicit solver (right) for a NACA0012 airfoil at $Re = 5000$ and $M = 0.5$, using a 3rd order SV scheme.
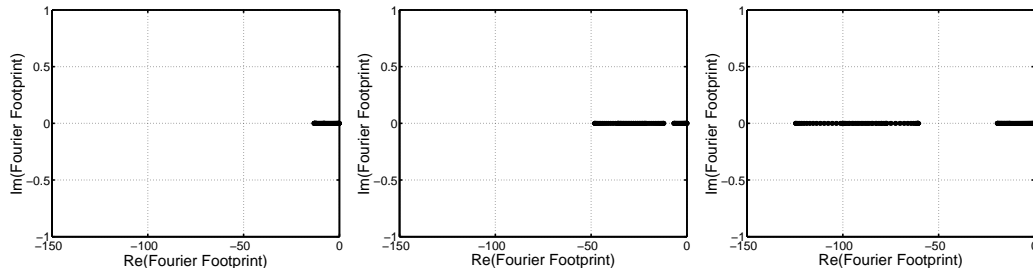

Figure 2: Fourier footprint for the 1D linear diffusion equation, corresponding to a second- (left), third- (middle) and fourth-order (right) SV scheme.

system of nonlinear equations at every iteration, which is linearized, and the resulting system of linear equations is inverted using a generalized minimal residual method. The histories of the mass density residual obtained with an explicit five-stage R-K solver and the implicit solver are shown in the middle and right plots of Figure 1. Clearly, the implicit solver needs far less iterations and is able to converge the solution much further than the explicit one. Moreover, the implicit solver is much more efficient in terms of CPU-time.

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