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POLYNOMIAL-CHAOS APPLIED TO LORENZ'S MODEL FOR QUANTIFICATION OF GROWTH OF INITIAL UNCERTAINTIES

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ABSTRACT

Model prediction of the evolution of a nonlinear system can be sensitive to errors in the initial condition. In fact, the accuracy of the prediction may deteriorate rapidly even if the initial condition is only slightly incorrect, as shown in the seminal paper by Lorenz [1], and the growth of the error can be so large as to render the prediction useless. To quantify such error growth in prediction is of practical interest; this is perhaps most evident in weather forecast, where the initial state of the weather is generally known only approximately from observations, and it becomes necessary to quantify the forecast error arising from the initial state to give confidence to the weather prediction. Lorenz's model [2], a strange attractor model, which is also a truncated weather model, has proven to be valuable for gaining insights into the problem of error growth; particularly, the model with its limited degrees of freedom is amenable to statistical analysis of prediction errors by means of Monte Carlo simulation (MC).

In this study, we address the problem of uncertainty growth in the Lorenz's model using the Polynomial Chaos (PC) approach [3], which treats the random error explicitly in the prognostic calculation. To the extent that uncertainty of the initial state is quantifiable in terms of random variables with some known probability distribution, the state variables are functions of such random variables, and thus are expressible in terms of, specifically, their polynomials. Lorenz's attractor model equations can be cast as deterministic equations for the coefficients of the polynomials of the random variables, and the forward integration of theses equations obtains coefficients' magnitudes that explicitly indicate the contribution of each random variable, as associated with each type of initial state error, to the uncertainty growth.

We show that for an initial state whose uncertainty obeys the Gaussian statistics and has a standard deviation less 1% of the norm of the initial state, a first order Hermite polynomial representation of the uncertainty growth reproduces virtually identically the growth given by MC; it may be noted that this growth is also reproducible with application of MC to the linear tangent form of the attractor equations. In short, when the error is small the random error associated with each state variable contributes independently to the growth of uncertainty; moreover, the error statistics remain Gaussian as the nonlinear attractor system evolves. The situation becomes more complex as the uncertainty grows to about 20%. At this point, the error statistics are no

longer Gaussian and the tangent linear equations lose validity; but, the second order Hermite polynomials which now contain interacting terms are able to obtain statistics that match those given by MC. Eventually the fifth order polynomials are needed to capture the correct statistics at 50% uncertainty level, and at this point the probability density in the state space takes on a ever changing complex geometry.

The fact that a low order PC representation can effectively reproduce MC results suggests that in the case of much more complex fluid system such as weather to which application of MC is impractical, PC may be used to estimate the uncertainty growth. In particular, when the error is small a simple application of the first order PC should be sufficient to determine the statistics of growing uncertainty, especially the direction of the fastest growth. Presently the determination of such direction has relied on the analysis of the tangent linear equations with results that vary with the end point of prognostic calculation and the choice of inner-product norm. Our study suggests that the error statistics and direction of the fastest growth can be determined continuously at all points of the prognostic calculation by using the PC approach, when MC may be impractical. Work sponsored by the Office of Naval Research.

REFERENCES

[1] E. Lorenz, "Deterministic nonperiodic flow", *J. Atmos. Sci.*, Vol. **20**, pp. 130-141, (1963).

[2] E. Lorenz, "Irregularity: a fundamental property of the atmosphere", *Tellus*, Vol. **36A**, pp. 98-110, (1984).

[3] O. Ghanem and P. Spanos, *Stochastic finite element: a spectral approach*, Springer, Berlin, 1991.