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# TOPOLOGICAL DERIVATIVE IN STEADY-STATE ORTHOTROPIC HEAT DIFFUSION PROBLEM

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## ABSTRACT

The topological derivative (Sokolowski and Zochowski, 1999; Céa *et al.* , 2000) allows us to quantify the sensitivity of the problem when the domain under consideration $\Omega$ is perturbed by introducing a hole, an inclusion or a source term in a small region $\mathcal{H}_\varepsilon$ of size $\varepsilon$ with center at an arbitrary point $\widehat{\mathbf{x}} \in \Omega$. Therefore, this derivative can be seen as a first order correction on a given shape functional $\psi(\Omega)$ to estimate $\psi(\Omega_\varepsilon)$, where $\Omega_\varepsilon$ is the perturbed domain. Thus, we have the following topological asymptotic expansion of $\psi$,

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f(\varepsilon) D_T(\widehat{\mathbf{x}}) + \mathcal{R}(f(\varepsilon)) \ , \tag{1}$$

where $f(\varepsilon)$ is a positive function that decreases monotonically such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0^+$, $\mathcal{R}(f(\varepsilon))$ contains all terms of higher order in $f(\varepsilon)$ and the term $D_T(\widehat{\mathbf{x}})$ is defined as the topological derivative of first order of $\psi(\Omega)$.

In the work of Sokolowski and Zochowski, 1999 (eq. 2.1 – problem $\mathcal{P}(\Omega_\rho)$), the topological sensitivity associated to the nucleation of a hole in an orthopic matrix was calculated. In order to simplify the analysis, the domain was perturbed considering an ellipse oriented in the directions of the orthotropy and with semi-axis proportinal to the material properties coefficients in each orthogonal direction. In this paper, we extend the above result considering as perturbation a circular hole with free boundary condition instead of an ellipse. More specifically, the shape functional is given by the total potential energy associated to the steady-state orthotropic heat diffusion problem, i.e.,

$$\psi(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} \mathbf{K} \nabla u_\varepsilon \cdot \nabla u_\varepsilon + \int_{\Gamma_N} \bar{q} u_\varepsilon \ . \tag{2}$$

with $u_\varepsilon$ solution of the following variational problem: find the temperature field $u_\varepsilon \in \mathcal{U}(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} \mathbf{K} \nabla u_\varepsilon \cdot \nabla \eta_\varepsilon + \int_{\Gamma_N} \bar{q} \eta_\varepsilon = 0 \qquad \forall \eta_\varepsilon \in \mathcal{V}(\Omega_\varepsilon), \tag{3}$$

where $\mathcal{U}(\Omega_\varepsilon)$ is the admissible functions set and $\mathcal{V}(\Omega_\varepsilon)$ is the admissible variations space, that are defined as

$$\mathcal{U}(\Omega_\varepsilon) = \{u_\varepsilon \in H^1(\Omega_\varepsilon) : u_\varepsilon|_{\Gamma_D} = \bar{u}\}, \quad \mathcal{V}(\Omega_\varepsilon) = \{\eta_\varepsilon \in H^1(\Omega_\varepsilon) : \eta_\varepsilon|_{\Gamma_D} = 0\}, \tag{4}$$

being $\Gamma_D$ and $\Gamma_N$ the Dirichlet and Neumann boundaries, such that $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$; $\bar{u}$ and $\bar{q}$ are the temperature and heat flux prescribed on $\Gamma_D$ and $\Gamma_N$, respectively; finally, $\mathbf{K}$ is the orthotropic thermal conductivity tensor.

Using the *topological shape-sensitivity method* proposed by Novotny *et al.* , 2003, and the polarization tensor (Pólya and Szegö, 1951) to obtain the asymptotic expansion for $u_\varepsilon$; we obtain the following structure for the topological derivative, taking $f(\varepsilon) = \pi\varepsilon^2$,

$$D_T(\widehat{\mathbf{x}}) = -\mathbf{T}\nabla u(\widehat{\mathbf{x}}) \cdot \nabla u(\widehat{\mathbf{x}}) \qquad \forall \widehat{\mathbf{x}} \in \Omega \ , \tag{5}$$

where $\mathbf{T} = \mathbf{T}(k_1, k_2)$ is a second order tensor that depends on the eigenvalues $k_1$ and $k_2$ of the conductivity tensor $\mathbf{K}$.

**Remark:** For isotropic material, $k_1 = k_2 = k$ and $\mathbf{T} = k\mathbf{I}$. Thus, the above result degenerates to the classical one, that is

$$D_T(\widehat{\mathbf{x}}) = -k\nabla u(\widehat{\mathbf{x}}) \cdot \nabla u(\widehat{\mathbf{x}}) \qquad \forall \widehat{\mathbf{x}} \in \Omega \ . \tag{6}$$

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