Finite Element Characteristics Old and New

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Abstract

Convection-diffusion-dissipation equations can be upwinded by a discretization of the total derivatives. Consider the solution u of $\partial_t u + a\nabla u = 0$ in $\Omega \times (0,T)$ with u^0 as initial condition with a, u^0 smooth and $a \cdot n = 0$ on the boundary.

$$u^{m+1}(x) = u^m(X(x)) \quad \text{with} \quad X(x) = x - a(x - a(x)\frac{\delta t}{2})\delta t \tag{1}$$

The scheme (1) is second order accurate. The difficulty then is to find a second order finite element approximation. Many research papers deal with this problem (see [5] and the references therein and more recently [3] [1] [6]) . The usual approach is to write the scheme in weak form and discretise by the finite element method. However all proofs require exact quadrature for the term containing $\bar{u}^{m+1}$ which in turn implies that the finite element mesh and the convected mesh are intersected. Since this is too costly we propose here an alternative based on an idea due to Yabe[7] which has been shown very accurate numerically but which has not yet been proven to work theoretically.

The simplest discretization in space is to appy the scheme at the center of gravity q_c^k of each triangle T_k: $u_h^{m+1}(x) = u_h^m(X(q_c^k)) \quad \forall x \in T_k \quad \forall k$ Scheme (**??**) is $O(\frac{h}{\delta t} + \delta t^2)$ and best at $\delta t = h^{1/3}$. To obtain a second order scheme in space, we intend to convect the function u and its gradients $v(x,t) = \nabla u(x,t)$. This leads to: $v^{m+1}(x) = \nabla X(x) v^m(X(x))$ which is second order accurate. Scheme (1) is discretized in space by a piecewise discontinuous linear or quadratic approximation on a triangulation $\mathcal{T}$:

$$u_h^{m+1}(x) = \nabla X(q_c)\nabla u_h^m(X(q_c))(x - q_c) + u_h^m(X(q_c)) \tag{2}$$

for all $x \in T$ the triangles of the triangulation; q_c denotes the center of T. The method is L^∞-stable and $O(\delta t^2 + \frac{h^{k+1}}{\delta t^2})$, with $k = 1, 2$ the degree of

the finite element approximation.

The rotating bell test case [5] has $a = (x, -y)^T$ and $u^0 = e^{-r|x-x_0|^2}$. Here the domain is truncated to the unit disk (the unit square when quadrilateral elements are used) and $r = 20, x^0 = (0.35, 0.35)^T$. At $t = 2\pi$ $u = u^0$ so the analytical solution is known.

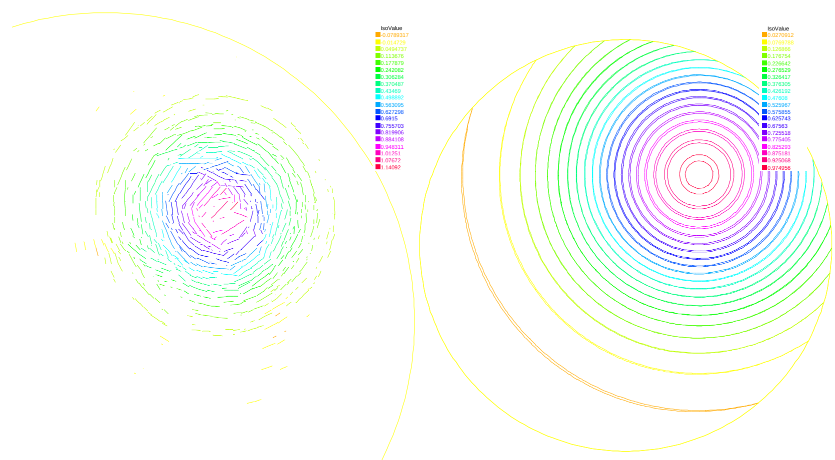

Figure 1: Left pure convection: zoom of the solution u_h computed with $P1$ discontinuous elements with a 5300 vertices and 50 time steps; there is no phase error and the exact solution is at the same position as the computed one . Right convection-diffusion: computed solution and exact (the perfect circles) solution; here the L^2 error is 0.009 after one turn.

References

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