**8th. World Congress on Computational Mechanics (WCCM8)**
**5th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)**
**June 30 –July 5, 2008**
**Venice, Italy**

# Identification of the Discontinuity in the Medium by Thermal Imaging

**Victor Isakov[1], Kyoungsun Kim[2] and * Gen Nakamura[3]**

[1] Department of Mathematics and Statistics, Wichita State University
Wichita, KS 67260-0033, USA
gnaka@math.sci.hokudai.ac.jp

[2] Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
kim@math.sci.hokudai.ac.jp

[3] Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
gnaka@math.sci.hokudai.ac.jp

**Key Words:** *Inverse boundary value problem, heat conductivity, unknown inclusion, thermal imaging.*

## ABSTRACT

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ $(1 \leq n \leq 3)$ with $C^2$ boundary if $n \geq 2$. We consider a heat conductor $\Omega$ with an inclusion $D$ such that $\overline{D} \subset \Omega$, $\Omega \backslash \overline{D}$ is connected, $\partial D$ is of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ if $n \geq 2$. Let the heat conductivity $\gamma(x)$ in $\Omega$ be given as follows:

$$\gamma(x) = \begin{cases} 1 & \text{for } x \in \Omega \setminus \overline{D} \\ k & \text{for } x \in D \end{cases} \tag{1}$$

with a positive constant $k$ which is not 1.

Now consider a boundary value problem to find a unique weak solution $u = u(f) \in H^{1,0}(\Omega_T)$ which satisfies

$$\begin{cases} \mathcal{P}_D u(x,t) := \partial_t u(x,t) - \mathrm{div}_x(\gamma(x,t)\nabla_x u(x,t)) = 0 \quad \text{in } \ \Omega_T := \Omega \times (0,T), \\ \partial_\nu u(x,t) = f(x,t) \quad \text{in } \ \partial\Omega_T, \quad u(x,0) = 0 \quad \text{for } \ x \in \Omega \end{cases} \tag{2}$$

for a given $f \in H^{1/2,0}(\partial\Omega_T)$.

It is well known that the boundary value problem (2) is well posed. That is there exists a unique solution $u = u(f) \in H^{1,0}(\Omega_T)$ to (2) and $u(f)$ depends continuously on $f \in H^{1/2,0}(\partial\Omega_T)$. Based on this, we define the *Neumann-to-Dirichlet map* $\Lambda_D$ as follows.

$$\Lambda_D : L^2((0,T); (H^{1/2}(\partial\Omega))^*) \to L^2((0,T); H^{1/2}(\partial\Omega))$$
$$f \mapsto u(f)|_{\partial\Omega_T}.$$

Now, we take the Neumann-to-Dirichlet map $\Lambda_D$ as measured data. Then, our inverse problem is to reconstruct the unknown inclusion $D$ from $\Lambda_D$.

For $(y,s),(y',s') \in \Omega_T \setminus \overline{D}$ such that $(y,s) \neq (y',s')$, let $\Gamma(x,t;y,s)$ and $\Gamma^*(x,t;y,s)$ be the fundamental solutions of $\mathcal{P}_\emptyset$ and $\mathcal{P}_\emptyset^*$, respectively. By Runge's approximation theorem given in [DKN], we can select two sequences of functions $\{v^j_{(y,s)}\}$ and $\{\varphi^j_{(y',s')}\}$ in $H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$ for arbitrary constant $\varepsilon > 0$ such that

$$\begin{cases} \mathcal{P}_\emptyset v^j_{(y,s)} = 0 & \text{in} \quad \Omega_{(-\varepsilon,T+\varepsilon)}, \\ v^j_{(y,s)}(x,t) = 0 & \text{if} \quad -\varepsilon < t \leq 0, \\ v^j_{(y,s)} \to \Gamma(\cdot,\cdot;y,s) & \text{in} \quad H^{2,1}(U) \quad \text{as} \quad j \to \infty, \end{cases}$$

and

$$\begin{cases} \mathcal{P}_\emptyset^* \varphi^j_{(y',s')} = 0 & \text{in} \quad \Omega_{(-\varepsilon,T+\varepsilon)}, \\ \varphi^j_{(y',s')}(x,t) = 0 & \text{if} \quad T \leq t < T+\varepsilon, \\ \varphi^j_{(y',s')} \to \Gamma^*(\cdot,\cdot;y',s') & \text{in} \quad H^{2,1}(U) \quad \text{as} \quad j \to \infty \end{cases}$$

for each open set $U$ in $\Omega_{(-\varepsilon,T+\varepsilon)}$ such that $\overline{U} \subset \Omega_{(-\varepsilon,T+\varepsilon)}$, $\Omega_{(-\varepsilon,T+\varepsilon)} \setminus \overline{U}$ is connected, $U$ has a Lipschitz boundary $\partial U$, and $\overline{U}$ does not contain $(y,s)$ and $(y',s')$. We call these $\{v^j_{(y,s)}\}, \{\varphi^j_{(y',s')}\}$ *Runge's approximation functions*.

**Definition 1.** ([DKN]) *Let $(y,s),(y',s') \in \Omega_T$ be such that $(y,s) \neq (y',s')$, and $\{v^j_{(y,s)}\}, \{\varphi^j_{(y',s')}\} \subset H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$ be Runge's approximation functions as above. Then, we define the pre-indicator function $I(y',s';y,s)$ as follows.*

$$I(y',s';y,s) = \lim_{j\to\infty} \int_{\partial\Omega_T} \left[ \partial_\nu v^j_{(y,s)}|_{\partial\Omega_T}\, \varphi^j_{(y',s')}|_{\partial\Omega_T} - \Lambda_D(\partial_\nu v^j_{(y,s)})|_{\partial\Omega_T}\, \partial_\nu \varphi^j_{(y's')}|_{\partial\Omega_T} \right]$$

*whenever the limit exists.*

**Definition 2.** *Let $C := \{c(\lambda)\,;\, 0 \leq \lambda \leq 1\}$ be a non-selfintersecting $C^1$ curve in $\overline{\Omega}$ which joins $c(0), c(1) \in \partial\Omega$ and $e(\lambda) := -\dot{c}(\lambda)/|\dot{c}(\lambda)|$. (We call this c a needle.) Then, for each $c(\lambda) \in \Omega_T$ and each fixed $s \in (0,T)$, we define the indicator function $J(c(\lambda),s)$ by*

$$J(c(\lambda),s) := \lim_{\epsilon\downarrow 0} \limsup_{\delta\downarrow 0} |I(c(\lambda-\delta)+\epsilon e(\lambda-\delta), s+\epsilon^2; c(\lambda-\delta), s)| \tag{3}$$

*whenever the limit exists.*

**Theorem 1.** *Let $C$ and $e(\lambda)$ be given as in Definition 2 above. Then, for a fixed $s \in (0,T)$, we have the followings.*

*(i) If the curve $C$ is in $\Omega \setminus \overline{D}$ except $c(0)$ and $c(1)$, then $J(c(\lambda),s) < \infty$ for all $\lambda$, $0 \leq \lambda \leq 1$.*

*(ii) Let $C \cap \overline{D} \neq \emptyset$ and $\lambda_s$ $(0 < \lambda_s < 1)$ be such that $c(\lambda_s) \in \partial D$, $c(\lambda) \in \Omega \setminus \overline{D}$ $(0 < \lambda < \lambda_s)$. Then,*

$$\lambda_s = \inf\{\, 0 < \lambda < 1\,;\, J(c(\lambda'),s) < \infty \quad \textit{for any}\ 0 < \lambda' < \lambda \,\}. \tag{4}$$

## REFERENCES


[DKN] Y. Daido, H. Kang, and G. Nakamura. "A probe method for the inverse boundary value problem of non-stationary heat equations". *Inverse Problem*, Vol **23**, 1787–1800, 2007.