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A DISCONTINUOUS GALERKIN METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS IN COMPLICATED DOMAINS

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ABSTRACT

Many practical applications require the solution of partial differential equations (PDEs) in complicated domains. We are especially interested in computing the flow around root networks of plants or in the pore space of porous media.

Classical numerical methods require a grid resolving the complicated geometry. Creating such grids is a highly involved process especially if coarse grids and high quality are required. Several methods have been developed that circumvent this problem. They are based on the use of a structured background mesh that encloses the domain. The Fictitious Domain method [GPP71] discretizes the PDE on the background mesh and adds the boundary conditions as additional constraints. This leads in general to a saddle point formulation that might be difficult to solve. The Composite Finite Element method [HS97] constructs piecewise linear basis functions on the fine background mesh und truncates them at the true boundary. Coarse grid basis functions for a geometric multigrid solver are constructed as combinations of fine grid basis functions. This method has been designed with emphasis on the fast solution of the arising linear system.

Our new method is based on the observation that in discontinuous Galerkin finite element methods the form of the element can be quite arbitrary. Thus the elements can be taken as the intersection of the structured background mesh with the complicated geometry. Assembling the stiffness matrix then requires integration over the interior and boundary of those non-standard elements. This is accomplished by constructing a local triangulation within each element. Note that the local triangulations of different elements are completely independent.

To be specific consider the following elliptic model problem in d space dimensions

$$\nabla \cdot j = f \text{ in } \Omega \subseteq \mathbb{R}^d \qquad j = -K\nabla p, \tag{1}$$

subject to boundary conditions

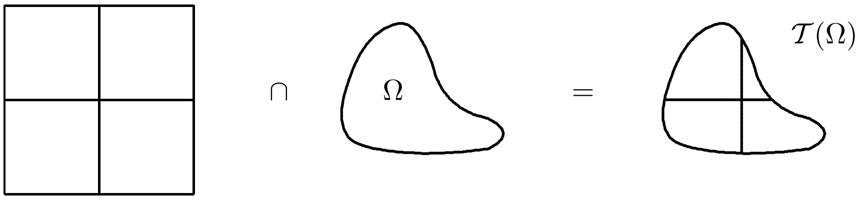
$$p = g \text{ on } \Gamma_D \subseteq \partial\Omega, \qquad j \cdot n = J \text{ on } \Gamma_N = \partial\Omega \setminus \Gamma_D. \tag{2}$$

We approximate the pressure p in the space of discontinuous finite element functions of order k

$$V_k = \{ v \in L_2(\Omega) \mid v|_E \in P_k, E \in \mathcal{T}(\Omega) \} \quad (3)$$

where $\mathcal{T}(\Omega) = \{E_1, \ldots, E_n\}$ is a partition of Ω into non-overlapping elements and P_k is the set of polynomials of at most degree k. $p_h \in V_k$ is determined using the discontinuous Galerkin method introduced by Oden, Babuška and Baumann in [OBB98].

The triangulation $\mathcal{T}(\Omega)$ used in the finite element algorithm is generated by intersecting a structured background mesh with the domain Ω as is indicated in the following figure:



Evaluation of the bilinear form and right hand side now require the computation of certain integrals over the interior and boundary of the non-standard elements. This is accomplished by constructing a local triangulation of each element. For more details we refer to [EB05].

In order to simplify the construction of the local triangulation, especially in three space dimensions, the geometry is represented by the level set of a scalar function on the (refined) background mesh. This is no restriction in our applications since the input is often given by image data from tomography.

In the talk we will present numerical results for solving the scalar elliptic model problem and Stokes equation in complicated domains in two and three space dimensions.

The implementation of the method is based on the DUNE framework [BDE+04].

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