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Bubble stabilized discontinuous Galerkin method using a symmetric formulation for second order elliptic problems. Part I: stability without penalty for the symmetric formulation.

*** Erik Burman[1], Benjamin Stamm[2]**

[1] Department of Mathematics
University of Sussex
Mantell Building, Falmer
BN7 9RF United Kingdom
E.N.Burman@sussex.ac.uk

[2] CMCS/IACS
Ecole Polytechnique Federale de Lausanne
CH-1015 Lausanne
benjamin.stamm@epfl.ch

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ABSTRACT

The discontinuous Galerkin method for second order elliptic problems dates back to the pioneering work of Baker [2], Arnold [1] and Wheeler [9] on the symmetric interior penalty method (SIPG) in the seventies. The original formulation common to all these references uses a symmetric bilinear form. Stability is obtained by adding a penalty term giving control on the inter-element solution jump. The bilinear form is coercive on the discrete space provided the stabilization parameter is sufficiently big. In the nineties Babuska, et al. [7] proposed a non-symmetric formulation (NIPG) which allowed for a less stiff penalty term. Indeed in this case the penalty parameter only has to be larger than zero in general. The non-symmetric discontinuous Galerkin method was given a full analysis in the work of Riviére et al. [8], Larson and Niklasson [6] and Brezzi and Marini [3]. In these references it was shown that the penalty term was necessary only for polynomial degrees larger than one. The non-symmetric form however suffers from the lack of adjoint consistency which makes the convergence rate of the error measured in the L^2-norm deteriorate.

To sum up: On the one hand we have the SIPG-method with optimal convergence properties but a stiff penalty term. On the other hand we have the NIPG-method with a less stiff penalty but without optimal convergence in the L^2-norm.

In a recent short communication Burman et al. [4] proved that, in the one dimensional case, for polynomial order $p \geq 2$, the SIPG method is stable and optimally convergent without penalty [4]. These results have been extended by Burman and Stamm [5] to the two and three dimensional case when using a finite element space consisting of piecewise affine functions, element wise enriched by a quadratic bubble. This leads to a discontinuous Galerkin method which does not need a penalty term without losing optimal convergence in the L^2-norm. A key feature of the method is that the gradient of the discrete solution is a Raviart-Thomas type function of lowest order locally on each element.

In this talk we will discuss the need of stabilization of the symmetric DG-formulation for second order elliptic problems. We show how the penalty can be eliminated without sacrificing control of the solution jumps in the one-dimensional case. For this method the matrix is no longer positive definite and the analysis relies on an inf-sup condition to recover control of the inter element jumps of the discrete solution. We then consider the multidimensional case and give an overview of known results for the bubble stabilized symmetric formulation for the elliptic problem.

In a special case we show the equivalence of our formulation and other approaches such as the classical mixed formulation using the Raviart-Thomas element or a Lagrange-multiplier approach. We will end the talk with a brief overview of applications that have been considered and discuss some open problems.

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