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On Γ-convergence of variational integrators for constrained systems

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ABSTRACT

For a physical system described by a motion in an energy landscape under holonomic constraints, the Γ-convergence of variational integrators to the corresponding continuum action functional and the convergence properties of solutions of the discrete Euler-Lagrange equations to stationary points of the continuum problem are studied [1]. This extends the results in [2] to constrained systems. The convergence result is illustrated with an example of flexible multibody dynamics.

Consider an n-dimensional mechanical system with the time-dependent configuration vector $\mathbf{q}(t) \in \mathcal{Q}$ and velocity vector $\dot{\mathbf{q}}(t) \in T_{\mathbf{q}(t)}\mathcal{Q}$, where $t \in [t_0, t_N] \subset \mathbb{R}$ denotes the time and $N \in \mathbb{N}$. Let the configuration be constrained by the function $\mathbf{g}(\mathbf{q}) = \mathbf{0} \in \mathbb{R}^m$ that restricts possible configurations to the constraint manifold $\mathcal{C} = \{\mathbf{q} \in \mathcal{Q} | \mathbf{g}(\mathbf{q}) = \mathbf{0}\}$. In the temporal discrete setting, using a constant time-step $h \in \mathbb{R}$, a path $\mathbf{q} : [t_0, t_N] \to \mathcal{Q}$ is replaced by a discrete path $\mathbf{q}_d : \{t_0, t_0+h, , t_0+Nh = t_N\} \to \mathcal{Q}$, where $\mathbf{q}_n = \mathbf{q}_d(t_0 + nh)$ is viewed as an approximation to $\mathbf{q}(t_0 + nh)$.

Let the action integral of the continuous Lagrangian $L : T_{\mathbf{q}(t)}\mathcal{Q} \to \mathbb{R}$ over one time interval be approximated by the discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ as it is standard in variational integrators [3]. Among various possible choices to approximate this integral, the midpoint rule is in use for the Lagrangian, i.e. $L_d(\mathbf{q}_n, \mathbf{q}_{n+1}) = hL\left(\frac{\mathbf{q}_{n+1}+\mathbf{q}_n}{2}, \frac{\mathbf{q}_{n+1}-\mathbf{q}_n}{h}\right)$. Let $L_d^{\mathcal{C}} = L_{d_{|\mathcal{C}\times\mathcal{C}}} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ denote the constrained discrete Lagrangian which restricts L_d to the constraint manifold. Then stationarity of the constrained discrete action

$$\delta S_d^{\mathcal{C}} = \delta \sum_{n=0}^{N-1} L_d^{\mathcal{C}}(\mathbf{q}_n, \mathbf{q}_{n+1}) = 0 \tag{1}$$

yields the constrained discrete Euler-Lagrange equations. The main results are the following two Theorems.

Theorem 1: Let $V \in C(\mathcal{Q}, \mathbb{R})$ with $|V(\mathbf{q})| \leq c(1 + |\mathbf{q}|^2)$. Then $S_d^{\mathcal{C}}(\cdot, (t_0, t_N))$ $\Gamma-$ converges in $L^2((t_0, t_N), \mathcal{Q})$ (on all open bounded time intervals (t_0, t_N)) and

$$\Gamma - \lim_{h\to 0} S_d^{\mathcal{C}}(\cdot, (t_0, t_N)) = S^{\mathcal{C}}(\cdot, (t_0, t_N)) \tag{2}$$

This states the $\Gamma-$convergence of the discrete action functional to the continuous one. As a second step, the convergence of stationary points of the discrete actions is considered.

Theorem 2: Let $V \in C(\mathcal{Q}, \mathbb{R})$ with $|V(\mathbf{q})| \leq c(1 + |\mathbf{q}|^2)$ and let $\mathbf{q}_d$ be a sequence of stationary points for $S_d^{\mathcal{C}}$ such that $|\mathbf{q}_d(t_0)|$ and $S_d^{\mathcal{C}}(\mathbf{q}_d, (t_0, t_N))$ are bounded uniformly. Then there exists a subsequence $\mathbf{q}_d \to \mathbf{q}$ in $L^p_{\text{loc}}(\mathbb{R}, \mathcal{Q})$ for all $1 \leq p < \infty$ and $\mathbf{q}$ is a stationary point of $S^{\mathcal{C}}$. Furthermore, $\dot{\mathbf{q}}_d \to \dot{\mathbf{q}}$ in $L^p_{\text{loc}}(\mathbb{R}, \mathcal{Q})$ for all $1 \leq p < \infty$.

The statements of Theorem 2 will be illustrated by means of a numerical example. It deals with a swing consisting of an elastic beam hinged at its ends to rigid bodies by revolute joints. The rigid bodies are fixed in space by spherical joints. An additional point mass is concentrated at the beams mid-point. The loading is a triangular pulse in longitudinal direction which is applied at the midspan mass. The present formulation of multibody dynamics relies on redundant coordinates subject to holonomic constraints. The constraints are treated using the discrete null space method which has been introduced in the framework of energy-momentum schemes in [4]. Figure 1 illustrates the motion and deformation of the swing, while the second order convergence of configurations and velocities can be observed from Figure 2.

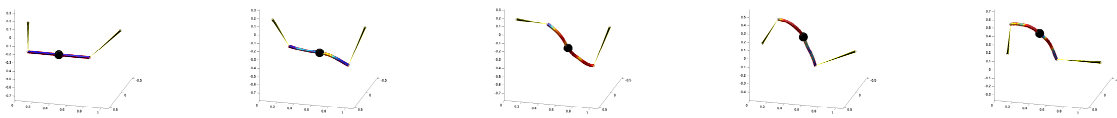

Figure 1: Three-bar swing: snapshots of the motion.

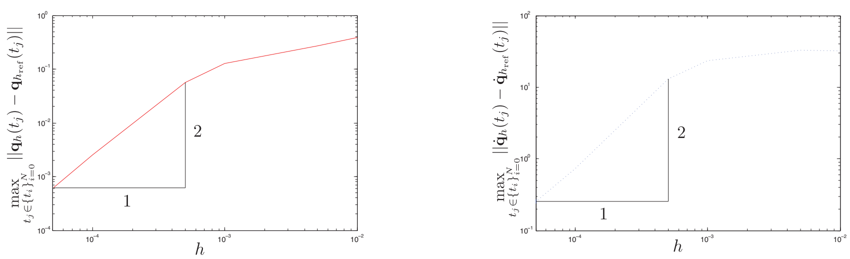

Figure 2: Three-bar swing: convergence of configuration and velocity to reference solution for $h \in [10^{-2} \ldots, 5 \cdot 10^{-5}]$.

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