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A REGULARIZED STABILIZED MIXED FEM FORMULATION FOR BINGHAM FLUIDS

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ABSTRACT

Viscoplasticity, as idealized by Bingham[1], is a phenomenon characterized by the existence of a residual value for the shear stress, beyond which the material would present a viscous flow. Defining τ_y as the residual stress, or yield stress, and μ as the plastic viscosity the first model for this behavior was

$$\tau(\mathbf{u}) = \tau_y + \mu\dot{\gamma} \Leftrightarrow \tau(\mathbf{u}) > \tau_y, \qquad \dot{\gamma}(\mathbf{u}) = 0 \Leftrightarrow \tau(\mathbf{u}) \leq \tau_y \tag{1}$$

System (1) presents a singularity, wich was handled by Glowinski[2] to a one variable problem by using a lagrangean multiplier and solving the formulation through a regularization method. Viscoplastic flow problems of incompressible fluids can be modelled by the following system of equations:

$$-\operatorname{div}(\tau(\mathbf{u})) + \nabla p = \mathbf{f} \text{ in } \Omega, \qquad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega$$

with $\mathbf{u} = \overline{\mathbf{u}}$ on $\partial\Omega$ and $\tau(\mathbf{u})$ is given by (1). Ω and $\partial\Omega$, are the domain and its boundary, respectively. Few numerical methods have been proposed for these problems and, in general, they transfer the instabilities to the boundaries, resulting in unstable pressure fields. Karam and Loula[3] proposed a mixed stabilized finite element formulation in velocity and discontinuous pressure variables able to handle the incompressibility constraint. Although obtaining stable results for linear case, when (1) is considered it is difficult to obtain theoretically the range of the stabilizing parameters.
Based on both approaches, in this work we propose a mixed regularized stabilized finite element method in velocity and discontinuous pressure to wich it is possible to obtain mathematically the range of stabilizing parameters.
Perturbing the classical Galerkin formulation by adding least-squares of the governing equations, stable regularized mixed finite element approximations for this problem are constructed here with descontinuous pressure interpolations as follows. Let $S_h^k(\Omega)$ be the finite element space of polynomials of degree k and class C^0, and $Q_h^l(\Omega)$ that of polynomials of degree l and class C^{-1}. Thus we can define the approximation spaces $V_h = (S_{0h}^k(\Omega))^2 = (S_h^k(\Omega) \cap H_0^1(\Omega)) \subset V, W_h = Q_h^l(\Omega) \subset L_0^2(\Omega)$. Then, the proposed formulation can be written in the following manner where the discontinuous pressure has been decomposed at the element level, to make the analysis easier, into a zero mean valued function, p_h^*, and a constant by part $\overline{p}_h$, that is, $p_h = p_h^* + \overline{p}_h, p_h^* \in W_h^*, \overline{p}_h \in \bar{W}_h$ such that $(W_h^* \subset W_h) = \{p_h^* \in L^2 : \int_{\Omega^e} p_h^* d\Omega = 0; \nabla p_h^e = \nabla p_h^{*e}\}; (\overline{W}_h \subset W_h) = \{\overline{p}_h \in L^2 : \nabla \overline{p}_h^e = 0, \overline{p}^e = \int_{\Omega^e} p^e d\Omega / \int_{\Omega^e} d\Omega\}$:

Problem PGG$_h$: *Find* $\{\mathbf{u_h}, p_h\} \in \mathbf{V_h} \times \mathbf{W_h}$, *such that*

$$A_h^*(\mathbf{u}_h, p_h^*; \mathbf{v}_h, q_h^*) + b(\overline{p}_h, \mathbf{v}_h) = F_h^*(\mathbf{v}_h, q_h^*), \quad \forall v_h \in \mathbf{V_h}, \forall q_h^* \in \mathbf{W_h^*}, \tag{2}$$

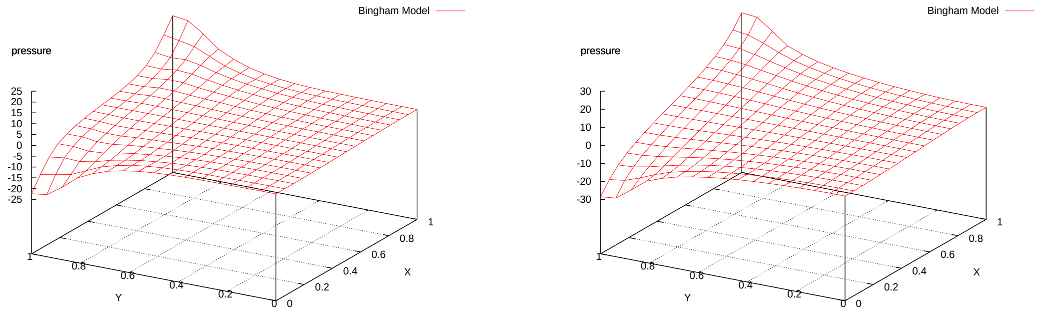
$$b(\overline{q}_h, \mathbf{u}_h) = 0, \forall \overline{q}_h \in \overline{W}_h, \tag{3}$$

$$\begin{aligned}
\textit{with } A_h^*(\mathbf{u}_h, p_h^*; \mathbf{v}_h, q_h^*) &= a(\mathbf{u_h}, \mathbf{v_h}) + j(\mathbf{v}_h) + b(p_h^*, \mathbf{v}_h) + b(q_h^*, \mathbf{v}_h) + \delta_2 2\mu(\operatorname{div}\mathbf{u_h}, \operatorname{div}\mathbf{v_h}) \\
&+ \frac{\delta_1 h^2}{2\mu}(-2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{u_h})) + \nabla p_h^*, -2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{v_h})) + \nabla q_h^*)_h \\
F_h^*(\mathbf{v}_h, q_h^*) = f(\mathbf{v_h}) &+ \frac{\delta_1 h^2}{2\mu}(\mathbf{f}, -2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{v_h})) + \nabla q_h^*)_h, \qquad b(\overline{q}_h, \mathbf{u}_h) = -(\overline{q}_h, \operatorname{div}\mathbf{u}_h) \\
a(\mathbf{u}_h, \mathbf{v}_h) &= 2\mu(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v}_h)), \qquad j(\mathbf{v}_h) = \tau_y \int_\Omega \frac{\boldsymbol{\epsilon}(\mathbf{u}_h) \cdot \boldsymbol{\epsilon}(\mathbf{v}_h)}{\sqrt{|\boldsymbol{\epsilon}(\mathbf{u}_h)|^2 + \eta^2}} d\Omega.
\end{aligned}$$

Where δ_1 and δ_2 are positive constants to be fixed Applying Brezzi's theorem we can prove existence, uniqueness and stability of **Problema PGG$_h$**, since we have the continuity of $A_h^*(\mathbf{u}_h, p_h^*; \mathbf{v}_h, q_h^*)$ and $b(p_h^*, \mathbf{v}_h)$, the LBB condition for $b(\overline{p_h}, \mathbf{v}_h)$ and the ellipticity of $A_h^*(\mathbf{u}_h, p_h^*; \mathbf{v}_h, q_h^*)$ obtained with the necessary condition:

$$\mathcal{R} = \delta_1 \rho \left(1 - \frac{(1/2)\delta_1 \gamma_1^2}{1 + 2\delta_1 \gamma_1^2 + \frac{\tau_y}{\mu\sqrt{C_2^2 + \eta^2}}} \right) - \frac{1}{\delta_2} > 0. \tag{4}$$

To confirm the performance of this formulation we present numerical results for the benchmark unit square cavity problem, with $\mathbf{u}(x, 1) = \{1, 0\}$ on $x \in [0, 1]$ and $\mathbf{u} = 0$ on the other boundaries. We adopted a regular mesh of 16 by 16 biquadratic elements of the same order for velocity and pressure. Fixing $\eta = 10^{-10}$, stable pressure elevations are shown in Figure 1 for $\tau_y = 1.0$ and 10.0. Stability was obtained here for a wide range of the δ_1 and δ_2 values and we present results for $\delta_1 = 5.0$ and $\delta_2 = 10.0$.



(a) (b)

Figure 1: Pressure elevations for $\delta_1 = 5.0$ and $\delta_2 = 10.0$ with (a)$\tau_y = 1.0$ and (b)$\tau_y = 10.0$.

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