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Model Adaptivity for Elasticity on Thin Domains

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ABSTRACT

The usage of a posteriori error estimation and adaptive procedures within the finite element method (FEM) is rather well-developed. The algorithms are usually based on the *discretization error*, i.e., the discrepancy between the continuous model—the exact solution of the differential equation—and the corresponding finite element approximation. However, it could be necessary to consider the choice of model carefully, as to keep the computational cost at a minimum (the most accurate model may be inherently expensive). Thus an appealing strategy would be to start with a simple model and increase its complexity only when needed.

We apply the concept of model adaptivity to the elasticity equation in two dimensions for thin domains. Typically, reduced models are then obtained using simplified deformation relations, e.g., such as the Bernoulli and Timoshenko beam theories. We shall instead follow Babuška, Lee and Schwab [1], and use a model hierarchy based on increasingly higher polynomial expansions through the thickness of the domain, coupled with a Galerkin approach. An *a posteriori* error indicator is introduced in terms of the total error, including discretization and modeling errors, which allows for adaptive refinement of the computational domain where the error is prominent, providing a means to obtain an inexpensive model meeting a prescribed accuracy.

The domain discretization is such that the mesh is resolved by several elements in one dimension, but it has no further elements through its thickness, in that sense making the domain thin. The total error, typically measured in energy norm, is then reduced by: either 1) introducing more elements; or 2) changing the underlying model. The latter is based on raising the order of the polynomial approximation along element sides (in the x_2-direction).

Consider the case of the exact solution residing in a tensor product space

$$\boldsymbol{u}(\boldsymbol{x}) = \big(\phi_1(x_1)\psi_1(x_2), \phi_2(x_1)\psi_2(x_2)\big) \in [V \times W]^2\,,$$

where $\phi_i \in V$ and $\psi_i \in W$. We now pose four different problems

1. find $\boldsymbol{u} \in [V \times W]^2$ such that $a(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v})$ for all $\boldsymbol{v} \in [V \times W]^2$,

2. find $\boldsymbol{u}^\psi \in \left[V^h \times W\right]^2$ such that $a(\boldsymbol{u}^\psi, \boldsymbol{v}) = L(\boldsymbol{v})$ for all $\boldsymbol{v} \in \left[V^h \times W\right]^2$,
3. find $\boldsymbol{u}^\phi \in \left[V \times W^h\right]^2$ such that $a(\boldsymbol{u}^\phi, \boldsymbol{v}) = L(\boldsymbol{v})$ for all $\boldsymbol{v} \in \left[V \times W^h\right]^2$,
4. find $\boldsymbol{u}^h \in \left[V^h \times W^h\right]^2$ such that $a(\boldsymbol{u}^h, \boldsymbol{v}) = L(\boldsymbol{v})$ for all $\boldsymbol{v} \in \left[V^h \times W^h\right]^2$,

and notice the energy orthogonality with respect to $\boldsymbol{v} \in \left[V^h \times W^h\right]^2$, that is, $a(\boldsymbol{u}^*, \boldsymbol{v}) = 0$, for $\boldsymbol{u}^*$ solving either of Problems (1–3). This property can be used to estimate

$$\|\boldsymbol{u} - \boldsymbol{u}^h\| := a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{u}^h)^{1/2}$$

in conjunction with Cauchy-Schwarz inequality and a saturation assumption:

$$\|\boldsymbol{u} - \boldsymbol{u}^h\| \leq \frac{1}{2(1-\beta)} \left(\|\boldsymbol{u}^\phi - \boldsymbol{u}^h\| + \|\boldsymbol{u}^\psi - \boldsymbol{u}^h\| \right), \quad \beta < 1. \tag{1}$$

The upper bound of (1) is the sum of the discretization and modeling errors, which also satisfy the following equalities

$$\|\boldsymbol{u}^\phi - \boldsymbol{u}^h\|^2 = L(\boldsymbol{u}^\phi - \pi_h \boldsymbol{u}^\phi) - a(\boldsymbol{u}^h, \boldsymbol{u}^\phi - \pi_h \boldsymbol{u}^\phi),$$
$$\|\boldsymbol{u}^\psi - \boldsymbol{u}^h\|^2 = L(\boldsymbol{u}^\psi - \pi_h \boldsymbol{u}^\psi) - a(\boldsymbol{u}^h, \boldsymbol{u}^\psi - \pi_h \boldsymbol{u}^\psi).$$

In practice $\boldsymbol{u}^* - \pi_h \boldsymbol{u}^*$ must be approximated, e.g., by substituting $\boldsymbol{u}^*$ with an approximation from a higher-resolution mesh (subject to either h-, p- or combined hp-refinement), which in turn is interpolated onto the coarser. By means of this example, indicators for the discretization and model errors, as well as an estimate of the total error, may be obtained.

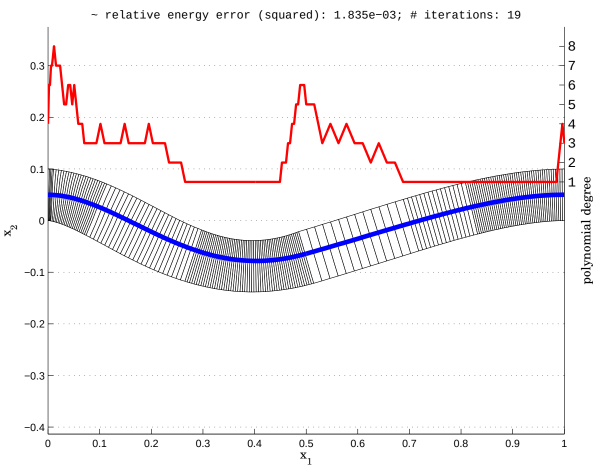

FIGURE 1: *The jagged solid line indicates model refinement at stress and strain extremes*

In Figure 1 a completely fixed console with piecewise Young's modulus reacts to a surface traction. The implemented algorithm, utilizing interpolatory Lagrangian basis functions, adaptively refined a two-element bilinear mesh to converge within a prescribed tolerance. (The solution of the corresponding Bernoulli beam equation has been enclosed within the console.)

REFERENCES

[1] I. BABUŠKA, I. LEE AND C. SCHWAB, *On the a posteriori estimation of the modeling error for the heat conduction in a plate and its use for adaptive hierarchical modeling*, Appl. Numer. Math. 14 (1994) 5–21.