8th World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30-July 5, 2008, Venice, Italy

A THREE DIMENTIONAL COLLISION WITH FRICTION

Shlomo Djerassi
Rafael, P.O.Box 2250 Haifa
e-mail: shlomod@rafael.co.il

Key Words: *3D Collision, Rough Collision, Coulomb's Friction, Slip Trajectory*

ABSTRACT

This work provides a comprehensive solution to the problem of 3D, one point collisions-with-friction of simple, non-holonomic systems. The underlined theory is based on Keller's idea [1], regarding the normal impulse as an independent integration variable. Keller generated differential equations with components of the relative velocity of the colliding points of two balls, as dependent variables, integrated these equations, and obtained the components of the relative velocity of separation, the normal and tangential impulses, and the changes in the motion variables. Stronge [2], Bhatt and Koechling [3], and Batlle [4] exploited this idea, analyzing certain special cases. They did not generalize their results to simple, non-holonomic systems, and did not examine the theory when used with Newton's, Poisson's and Stronge's hypotheses. It is the purpose of this work to fill these gapes.
The point of departure for the present work is the *p*-dimensional governing (matrix) equation

$$\left|\Delta u_1 \;...\; \Delta u_p\right|^T = -I_n\mathbf{M}^{-1}\left|\mathbf{v}_1^R\cdot\mathbf{n} \;...\; \mathbf{v}_p^R\cdot\mathbf{n}\right|^T - I_t\mathbf{M}^{-1}\left|\mathbf{v}_1^R\cdot\mathbf{t}\;...\;\mathbf{v}_p^R\cdot\mathbf{t}\right|^T - I_s\mathbf{M}^{-1}\left|\mathbf{v}_1^R\cdot\mathbf{s}\;...\;\mathbf{v}_p^R\cdot\mathbf{s}\right|^T \quad (1)$$

of a p DOF simple, nonholonomic system, where $\mathbf{M}^{-1}$ is the inverse of the mass matrix, $\mathbf{v}_1^R$ ... $\mathbf{v}_p^R$ are coefficients of the motion variables in the expression of the relative velocity of the colliding points, and **n**, **t** and **s** are dextral, mutually perpendicular unit vectors, **n** being normal to the 'plane of collision'. This equation is obtained by the integration of the system equations of motion between t_1 and t_2, the collision starting and terminating instants; and possess p+3 unknowns I_n, I_t and I_s, the normal and tangential impulses, and $\Delta u_1,...,\Delta u_p$, the changes in the motion variables. It can be shown that this equation can be replaced with

$$\mathrm{d}\tilde{v}_n = m_{nn}\mathrm{d}\tilde{I}_n + m_{nt}\mathrm{d}\tilde{I}_t + m_{ns}\mathrm{d}\tilde{I}_s \quad (2)$$

$$\mathrm{d}\tilde{v}_t = m_{nt}\mathrm{d}\tilde{I}_n + m_{tt}\mathrm{d}\tilde{I}_t + m_{ts}\mathrm{d}\tilde{I}_s \quad (3)$$

$$\mathrm{d}\tilde{v}_s = m_{ns}\mathrm{d}\tilde{I}_n + m_{ts}\mathrm{d}\tilde{I}_t + m_{ss}\mathrm{d}\tilde{I}_s \quad (4)$$

where $m_{nn}, m_{nt}, ..., m_{ss}$ are configuration dependent parameters, $\mathrm{d}\tilde{v}_n$, $\mathrm{d}\tilde{v}_t$ and $\mathrm{d}\tilde{v}_s$ are the differentials of the relative normal ($\tilde{v}_n$) and tangential ($\tilde{v}_t, \tilde{v}_s$) velocities, and $\mathrm{d}\tilde{I}_n$, $\mathrm{d}\tilde{I}_t$, and $\mathrm{d}\tilde{I}_s$ denote differentials of $\tilde{I}_n$, $\tilde{I}_t$ and $\tilde{I}_s$, where $I_n = \tilde{I}_n(t_2)$, $v_n = \tilde{v}_n(t_2)$, etc.
Following Keller [1], let s (the 'slip speed') and f (the 'slip orientation') be defined

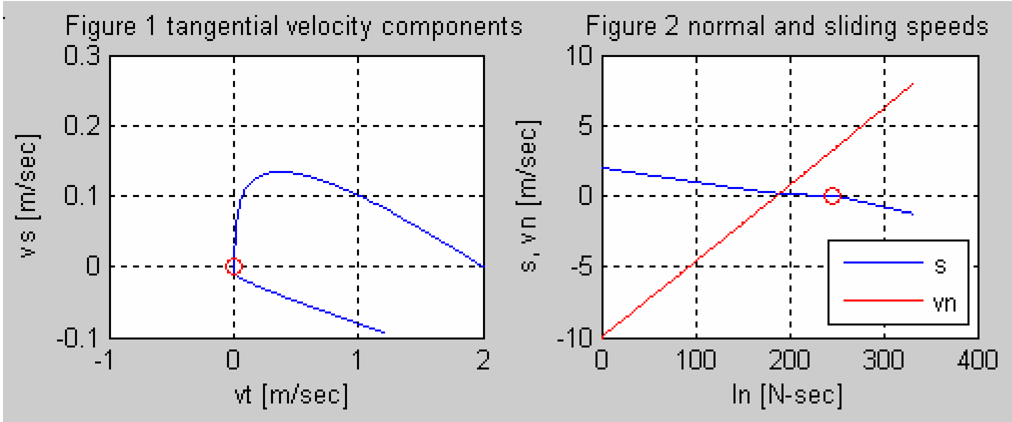
$$\tilde{v}_t = sc\mathrm{f},\; \tilde{v}_s = ss\mathrm{f} \;\Rightarrow\; \mathrm{d}\tilde{v}_t = c\mathrm{f}\,\mathrm{d}s - ss\mathrm{f}\,\mathrm{d}\mathrm{f},\; \mathrm{d}\tilde{v}_s = s\mathrm{f}\,\mathrm{d}s + sc\mathrm{f}\,\mathrm{d}\mathrm{f} \quad (5)$$

(see also [2]) and note that as long as there is sliding

$$\mathrm{d}\tilde{I}_t = -\mathrm{m}\mathrm{d}\tilde{I}_n c\mathrm{f},\; \mathrm{d}\tilde{I}_s = -\mathrm{m}\mathrm{d}\tilde{I}_n s\mathrm{f}, \quad (6)$$

where m is Coulmb's coefficient of friction. With *f*, *g* and *h* defined as

$$f \quad \mathrm{d}\tilde{v}_n/\mathrm{d}\tilde{I}_n,\; g \quad \mathrm{d}\tilde{v}_t/\mathrm{d}\tilde{I}_n c\mathrm{f} + \mathrm{d}\tilde{v}_s/\mathrm{d}\tilde{I}_n s\mathrm{f},$$
$$h \quad -\mathrm{d}\tilde{v}_t/\mathrm{d}\tilde{I}_n s\mathrm{f} + \mathrm{d}\tilde{v}_s/\mathrm{d}\tilde{I}_n c\mathrm{f} \quad (7)$$



it can be shown, with the aid of Eqs. (2)-(7), that

$$f = m_{nn} - \mu m_{nt} c\phi - \mu m_{ns} s\phi \tag{8}$$

$$g = m_{nt} c\phi + m_{ns} s\phi - \mu m_{tt} c^2\phi - \mu m_{ss} s^2\phi - 2\mu m_{st} s\phi\, c\phi \tag{9}$$

$$h = -m_{nt} s\phi + m_{ns} c\phi - \mu m_{st}(c^2\phi - s^2\phi) + \mu(m_{tt} - m_{ss}) s\phi\, c\phi\ . \tag{10}$$

and that Eqs. (2)-(4) and (6) can be replaced with the following set of differential equations:

$$\begin{aligned} &\mathrm{d}s / \mathrm{d}\tilde{I}_n = g,\ \ \mathrm{d}\phi / \mathrm{d}\tilde{I}_n = h / s,\ \mathrm{d}\tilde{v}_n / \mathrm{d}\tilde{I}_n = f, \\ &\mathrm{d}\tilde{I}_t / \mathrm{d}\tilde{I}_n = -\mu \cos\phi,\ \mathrm{d}\tilde{I}_s / \mathrm{d}\tilde{I}_n = -\mu \sin\phi . \end{aligned} \tag{11}$$

Equations (11) are used to uncover s, ϕ, $\tilde{v}_n$, $\tilde{I}_t$ and $\tilde{I}_s$ as functions of $\tilde{I}_n$ for the sliding portion of the collision, and, when applied to a flying beam hitting a bump (the exact details are given in the paper), give rise to figures such as 1 and 2. If s reduces to zero (see Fig. 2), then sliding halts, an event followed either by sticking or by sliding resumption (see [2], p. 68). To uncover which of these occur, assume first that sticking follows, in which case Eqs. (2)-(4) are rewritten with $\tilde{v}_t \equiv 0$ and $\tilde{v}_s \equiv 0$. The resulting equations become algebraic, with differentials replaced by differences; and can be solved as follows,

$$I_n = \tilde{I}_n^{ST} + \Delta\tilde{v}_n c_n \quad I_t = \tilde{I}_t^{ST} + \Delta\tilde{v}_n c_t \quad I_s = \tilde{I}_s^{ST} + \Delta\tilde{v}_n c_s \tag{12}$$

where $\tilde{I}_n^{ST}$ is the value of $\tilde{I}_n$ when sticking firstly occurs (circles in Figs. 1 and 2), $\Delta\tilde{v}_n = v_n^S - v_n^{ST}$, and c_n, c_t and c_s are functions of $m_{nn}, m_{nt}, \ldots, m_{ss}$. It is next argued that if $\mu_s < \mu$, where $\mu_s \doteq I_n/(I_t^2 + I_s^2)^{1/2} = c_n/(c_t^2 + c_s^2)^{1/2}$, then sticking prevails. Otherwise, sliding is resumed. It is finally shown in the paper that the introduction Newton's [6], Poisson's [5] and Stronge's [2] hypotheses underlie theories for the identification of the type of collision and the evaluation of I_n, I_t and I_s, and ultimately of $\Delta u_1, \ldots, \Delta u_p$ (Eq. (1)). These theories and the associated algorithms are described in the paper, their generality is discussed, as is their energy-consistency. Finally, the paper investigates the manner the 3D theories tie up with their respective 2D counterparts.

REFERENCES

[1] Keller, J. B.: Impact with Friction; J. of Appl. Mech. **53**, 1-4 (1986)
[2] Stronge, W. J., *Impact Mechanics*, Cambridge University Press, Cambridge, 2000
[3] Bhatt, V. and Koechling, J.: Three-Dimensional Frictional Rigid-Body Impact. J. of Appl. Mech. **62**, 893-898 (1995)
[4] Battle, J. A.: The Sliding Velocities Flow of Rough Collisions in Multibody Systems. J. of Appl. Mech. **63**, 804-809 (1996)
[5] Poisson, S. D., *Mechanics*, Longmans, London, 1817
[6] Newton, I., *Philosophias Naturalis Principia Mathematica*, R. Soc. Press, London, 1686