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Implementing slip boundary conditions on curved boundaries

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ABSTRACT

At the interface between a fluid and a bordering solid structure, it is generally assumed that the fluid satisfies the no-slip boundary condition. This assumption is being more and more questioned in the case of polymeric fluids and even for Newtonian fluids, particularly in highly confined geometries like capillaries or in microfluidic devices. In these cases, thanks to new and more sophisticated measurement techniques, it is now well admitted that fluid slip over solid surfaces may occur [1].

While the finite element approximation of Stokes or Navier-Stokes equations with no-slip boundary conditions has been largely analyzed and experimented computationally, slip boundary conditions have paid much less attention. The first theoretical results in this direction were provided by Verfürth [2-4], for Navier-Stokes equations with the *free* slip boundary condition (written here in 2D)

$$\boldsymbol{u} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma, \tag{1}$$

$$\boldsymbol{\nu} \cdot \boldsymbol{T}(\boldsymbol{u}, p) \cdot \boldsymbol{\tau} = 0, \quad \text{on } \Gamma. \tag{2}$$

Here Γ is the solid surface where slip occurs, $\boldsymbol{u}$ and p stand for velocity and pressure respectively, $\boldsymbol{T}(\boldsymbol{u}, p)$ is the stress tensor, $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are the outward unit normal vector and a tangent unit vector along Γ respectively.

Different weak formulations have been devised to form the basis of a finite element approximation of Stokes or Navier Stokes equations with this boundary condition. They differ primarily in the treatment of the boundary flux condition (1), while (2) is a *natural* boundary condition which is always treated in the usual way without any difficulty.

In this talk we shall present our numerical investigations with three different strategies which differs according to the way to impose the flux at the boundary : *(i)* imposing (1) in a strong sense; *(ii)* using the Lagrange multiplier method; *(iii)* or using Nitsche's method [5]. While our numerical results confirm

the theoretical convergence results in [2-4,6-8] in cases where Γ is flat, the situation is by far less satisfactory in cases where Γ has a non-vanishing curvature.

The finite elements that are tested are defined on a mesh built over a polygonal domain approaching the (hence non-polygonal) domain with curved boundary Γ. As a result, if we denote by h the corresponding mesh size, the polygonal domain Ω_h induces an outward unit normal $\boldsymbol{\nu}_h$ defined on Γ_h. We shall show that we obtain convergent or divergent finite element approximations, not only depending on which one of the three strategies listed above is chosen, but also depending on whether we use $\boldsymbol{\nu}$, $\boldsymbol{\nu}_h$ or other alternative substitutions to $\boldsymbol{\nu}_h$ in the finite element formulations of the problem. The paradoxical divergence results are connected to Babuska's paradox observed for a plate equation posed on a circular domain with simple support boundary conditions [9].

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