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A SEMISMOOTH NEWTON METHOD FOR INVERSE PROBLEMS WITH SPARSITY CONSTRAINTS

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ABSTRACT

In this work we consider the optimization problem

$$\text{Minimize}\quad \frac{1}{2}\|Ku-f\|_2^2+\sum_{k=1}^{\infty} w_k\,|u_k| \quad \text{over } u\in\ell^2. \tag{1}$$

Here, $K:\ell^2\to X$ is a linear and injective operator mapping the sequence space ℓ^2 into a Hilbert space X, $f\in X$ and $w=\{w_k\}$ is a sequence satisfying $w_k\geq w_0>0$.

One well understood algorithm for the solution of (1) is the so-called iterated soft-thresholding for which convergence has been proven in [4], see also [2]. While the iterated soft-thresholding is easy to implement it converges slow in practice (in fact the method converges linearly but with a constant close to one [2]). Another well analyzed method is the iterated hard-thresholding which converges like $\mathcal{O}(n^{-1/2})$ [1] (i.e. even slower than the iterated soft-thresholding but practically it is faster in many cases).

In this article we derive an algorithm for which we prove local superlinear convergence in the infinite dimensional setting. Our algorithm is an active set, or semismooth Newton, method and hence, the analysis is based on the notion of slant differentiability [3]. The semismooth Newton method is easily implementable as an active set method.

The semismooth Newton method build up on the simple fact that minimizers of the functional (1) are characterized by

$$\bar u=\mathbf{S}_{\gamma w}(\bar u-\gamma K^*(K\bar u-f))\quad \text{for any } \gamma>0 \tag{2}$$

where the *soft-thresholding of* u *with the sequence* w is defined as

$$\mathbf{S}_w(u)_k=\max\{0,|u_k|-w_k\}\operatorname{sgn}(u_k).$$

It can be shown that the equation (2) is semismooth (or slantly differentiable), and hence, Newton's method may be applied.

For $u \in \ell^2$, the active set $\mathring{A}(u)$ and the inactive set $I(u)$ are given by

$$
\begin{aligned}
A(u) &= \{k \in \mathbf{N} : |u - \gamma K^*(Ku - f)|_k > \gamma\, w_k\} \\
I(u) &= \{k \in \mathbf{N} : |u - \gamma K^*(Ku - f)|_k \le \gamma\, w_k\}.
\end{aligned}
$$

Whenever the active and inactive sets correspond to an iterate u^n, they will be denoted by A_n and I_n, respectively.

The mapping $F : \ell^2 \to \ell^2$,

$$F(u) = u - \mathbf{S}_{\gamma w}(u - \gamma K^*(Ku - f))$$

is Newton differentiable. We split the operator K^*K according to

$$K^*K = \begin{pmatrix} M_{AA} & M_{AI} \\ M_{IA} & M_{II} \end{pmatrix}.$$

Then a slant derivative is given by

$$G(u) = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_I \end{pmatrix} + \begin{pmatrix} \mathrm{Id}_A & 0 \\ 0 & 0 \end{pmatrix} (\gamma K^*K) = \begin{pmatrix} \gamma M_{AA} & \gamma M_{AI} \\ 0 & \mathrm{Id}_I \end{pmatrix}.$$

The semismooth Newton method

$$u^{n+1} = u^n - G(u^n)^{-1} F(u^n)$$

can be implemented as an active set strategy where in each step just a (usually small) finite system of linear equations has to be solved, see [5] for details. Moreover, local superlinear convergence of the algorithm is proven in [5]. Numerical examples in [5] indicate, that the algorithm compares favorable with existing methods.

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