8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineeering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

The Multidimensional Refinement Indicators Algorithm for Adaptive Parameterization

H. Ben Ameur[1], * F. Clément[2], P. Weis[2] and G. Chavent[2]

[1] FSB and ENIT-LAMSIN
University of Tunis, BP 37, Le Belvdre
1002 Tunis, Tunisia
hend.benameur@enit.rnu.tn

[2] Projet Estime, INRIA-Rocquencourt
Domaine de Voluceau, BP 105, 78153 Le Chesnay cedex, France
Francois.Clement@inria.fr
Pierre.Weis@inria.fr
Guy.Chavent@inria.fr

Key Words: *Inverse problem, parameter estimation, adaptive parameterization, image segmentation.*

ABSTRACT

The estimation of distributed parameters in a partial differential equation (PDE) from measures of the solution of the PDE may lead to underdetermination problems. The choice of a parameterization is a frequently used way of adding a priori information by reducing the number of unknowns according to the physics of the problem. The refinement indicators algorithm proposed in [2, 1] for the estimation of hydraulic transmissivities provides a fruitful adaptive parameterization technique that parsimoniously opens the degrees of freedom in an iterative way driven at first order by the model to locate the discontinuities of the sought parameter.

The main findings are:
(i) The generalization of the refinement indicators algorithm to the estimation of distributed *multidimensional parameters in any PDE*.
(ii) The quantitative relationship between the refinement indicator and the decrease of the least-squares data misfit objective function in the linear case.
(iii) An amazing image segmentation technique corresponding to the application of the multidimensional algorithm to the identity model in the RGB color space.

We consider the case where parameters are distributed and possibly vector valued, i.e. belong to a space P of functions defined over a domain Ω with values in $\mathbb{R}^{n_p}$ ($n_p \geq 1$ is the dimension of the vector parameter $p(x)$ at any $x \in \Omega$). Let the forward operator $\mathcal{F}$ be the composition of the model operator computing the solution of the PDE for a given parameter p with the observation operator computing the output of the observation device applied to the solution of the PDE. Let $d \simeq \mathcal{F}(p_{\rm true})$ be some measurements of the output for some unknown parameter $p_{\rm true} \in P$, and let

$$\mathcal{J}(p) = \frac{1}{2}\|d - \mathcal{F}(p)\|^2 \tag{1}$$

be the least-squares misfit between the data d and the corresponding quantities $\mathcal{F}(p)$ computed from the current parameter p. The unknown parameter $p_{\rm true}$ can be determined by solving the least-squares inverse problem set as the minimization of the misfit $\mathcal{J}(p)$ with respect to the parameter $p \in P^{\rm ad}$.

To reduce the number of unknowns, we search for the parameter in a subspace of P of (small) finite dimension. More precisely, a sequence of subspaces $(P_n)_n$ is constructed and at each iteration step an approximation $p_n \in P_n$ of the unknown parameter is computed. The idea of adaptive parameterization is to incrementally add degrees of freedom. These degrees of freedom correspond to the discontinuities of the sought parameter, they are chosen according to *refinement indicators*.

It is convenient to consider P_n as the range of an—unknown—*parameterization map*

$$\mathcal{P}_n : m_n \in M_n^{\mathrm{ad}} \longmapsto p_n \in P_n^{\mathrm{ad}} \tag{2}$$

where m_n is the *coarse* parameter (of small finite dimension), by opposition to the *fine* parameter p_n (of large, and possibly infinite dimension). M_n^{ad} is the space of admissible coarse parameters. Typically, m_n is made of the coefficients of $p_n \in P_n$ on a basis of P_n, in which case $\mathcal{P}_n$ is a linear operator. But the parameterization map can also be nonlinear. For any parameterization map $\mathcal{P}_n$, we define the same objective function on M_n^{ad} by

$$\mathcal{J}_n(m_n) = \mathcal{J}(\mathcal{P}_n(m_n)), \tag{3}$$

and the least-squares problem becomes:

$$\text{minimize } \mathcal{J}_n(m_n) \text{ for } m_n \in M_n^{\mathrm{ad}}. \tag{4}$$

Going from the nth iteration to the next one, we refine one zone of the current parameterization, which means we allow the parameter to have a discontinuity in this zone at some location. The norm of the derivative of the objective function at the optimum with respect to the amplitude c of the discontinuity taken at $c = 0$ gives us the first order effect on the optimal value of the objective function produced by the refinement. This norm is the *refinement indicator* corresponding to the discontinuity. This is also the norm of the Lagrange multiplier associated with the constraint expressing the discontinuity jump.

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