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A VARIATIONAL APPROACH TO SOLVE CAUCHY PROBLEM FOR STEADY STATE STOKES FLOW

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ABSTRACT

Among the applications of inverse methods to fluid mechanics, one may think of the characterization of physical parameters like shear viscosity in rheology or optimal control in the field of drag reduction. But, the identification of boundary condition on an unaccessible boundaries (data completion), is another inverse problem relevant to fluid mechanics. The data completion in fluid mechanics gathers several differents applications like, for exemple, measurements in experimental fluid mechanics, i.e provide measurements in regions of the flow where probing is unreachable or to complete experimental data by virtual numerical probing. Data completion of large scale models like ocean and atmosphere feeded, with scattered measurements is yet another possibility.

In this work, the objective is to extend a variational method based on the minimisation of a error functional, by applying it to fluid mechanics problems and specifically to low velocity flows, i.e. Stokes flows.

Previous works concerning data completion of boundary data may be distinguished into two areas: On one side, an alternating method, usually named Kozlov method, has been applied to the Stokes equations in a theorical framework by Bastey et al. [6] and Johansson [7]. On the other side, a variational method based on the minimisation of an error functional, proposed initially by Kohn et al. [5] in tomography, has been extended to linear elasticity problems by Andrieux et al. [1, 2] and Baranger et al. [3]. Recently, a further extension to Darcy model have been put forward by Escriva et al. [4]. The current work appears as intermediate step between the solving of the Cauchy problem on Darcy model and the full Navier-Stokes model.

Cauchy problem for the steady Stokes equations is derived from the linearisation of the steady incompressible Navier-Stokes equation for a newtonian fluid with constant viscosity μ. It can be stated as follows: for all compatible pairs data $(\mathbf{u}_m, \mathbf{t}_m)$, find $(\mathbf{u}_u, \mathbf{t}_u)$ on Γ_u such that:

$$\begin{cases} \mu\Delta\mathbf{u} - \boldsymbol{\nabla} p = \mathbf{f} & \text{in} \quad \Omega, \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{u}_m, \;\; \boldsymbol{\sigma}\mathbf{n} = \mathbf{t}_m & \text{on} \quad \Gamma_m, \;\; \text{and } \mathbf{u} = \mathbf{u}_u, \;\; \boldsymbol{\sigma}\mathbf{n} = \mathbf{t}_u \;\; \text{on} \;\; \Gamma_u \end{cases} \tag{1}$$

where $\mathbf{u}$ is the fluid velocity field, p the static pressure, $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{d}$ the stress tensor and $\mathbf{d} = \frac{1}{2}\left(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}\right)$ the strain tensor. We denote by Γ_m the boundary where the data are overspecified and by Γ_u the boundary where the data are unknown. The vectors $\mathbf{u}_m$ and $\mathbf{t}_m$ stand for the measured velocity and the measured stress vector, $\mathbf{u}_u$ and $\mathbf{t}_u$ are respectively the unknown velocity and stress vectors.
To solve this Cauchy problem, we propose to reformulate it into two well-posed problems:

$$\left\{\begin{array}{lll} \mu\Delta\mathbf{u}_1 - \boldsymbol{\nabla} p_1 = \mathbf{f} & \text{in} & \Omega, \\ \boldsymbol{\nabla}\cdot\mathbf{u}_1 = 0 & \text{in} & \Omega, \\ \mathbf{u}_1 = \mathbf{u}_m & \text{on} & \Gamma_m \\ -p_1\mathbf{n} + 2\mu\boldsymbol{d}(\mathbf{u}_1)\,\mathbf{n} = \boldsymbol{\eta} & \text{on} & \Gamma_u \end{array}\right. \qquad \left\{\begin{array}{lll} \mu\Delta\mathbf{u}_2 - \boldsymbol{\nabla} p_2 = \mathbf{f} & \text{in} & \Omega, \\ \boldsymbol{\nabla}\cdot\mathbf{u}_2 = 0 & \text{in} & \Omega, \\ \mathbf{u}_2 = \boldsymbol{\tau} & \text{on} & \Gamma_u \\ -p_2\mathbf{n} + 2\mu\boldsymbol{d}(\mathbf{u}_2)\,\mathbf{n} = \mathbf{t}_m & \text{on} & \Gamma_m \end{array}\right. \tag{2}$$

Hence, an error functional can be defined as follows:

$$E(\boldsymbol{\eta},\boldsymbol{\tau}) = \int_{\Gamma_u}(\boldsymbol{\tau} - \boldsymbol{u}_1)\cdot(\boldsymbol{\sigma}(p_2,\boldsymbol{u}_2)\boldsymbol{n} - \boldsymbol{\eta})) + \int_{\Gamma_m}(\boldsymbol{u}_2 - \mathbf{u}_m)\cdot(\mathbf{t}_m - \boldsymbol{\sigma}(p_1,\boldsymbol{u}_1)\boldsymbol{n})) \tag{3}$$

The functional E is always positive and expresses an energy-like error between the two fields ($\boldsymbol{u}_1$,p_1) and ($\boldsymbol{u}_2$,p_2). Assuming that the data $\mathbf{u}_m$ and $\mathbf{t}_m$ are compatible, the functional E vanishes when the pair $(\boldsymbol{\eta},\boldsymbol{\tau})$ meets the real data $(\mathbf{u}_u,\mathbf{t}_u)$. The boundary condition identification problem is then formulated as follows:

$$(\mathbf{u}_u,\mathbf{t}_u) = \arg\min_{(\boldsymbol{\eta},\boldsymbol{\tau})} E(\boldsymbol{\eta},\boldsymbol{\tau})$$

Results will be discussed and illustrated on an application. It consist of the recovery of the stress on a static inner cynlinder of a Taylor-Couette, setup in a annular 2D geometry with overspecified velocity and stress on the rotating outer cylinder.

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