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A CONSTITUTIVE FORMULATION OF NONLOCAL PLASTICITY

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Abstract. In the present paper a nonlocal plasticity model is analyzed. Elasticity is considered to be governed by local forces, so that only the dissipation processes are nonlocal (see, e.g., Pijaudier-Cabot and Bazant [1] and Bazant and Lin [2]). Differing from other proposed models (see, e.g., Borino *et al.* [3] and Marotti de Sciarra [4]) in which the isotropic hardening/softening variables are considered as nonlocal, in the present paper the nonlocality is extended in order to include the kinematic hardening behaviour as well, so that both types of hardening (kinematic and isotropic) are considered as nonlocal. The present formulation satisfies a variational condition representing nonlocal maximum plastic dissipation. The proposed constitutive formulation of nonlocal plasticity is thus equipped with a sound variational basis.

1. THE NONLOCAL MODEL

We denote with $\boldsymbol{\sigma} : \mathit{\Omega} \to \mathcal{S}$ the stress tensor. An additive decomposition of the total strain $\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} : \mathit{\Omega} \to \mathcal{D}$ into an elastic part and a plastic part is assumed, so that $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$. The kinematic internal variable $\boldsymbol{\alpha}_1 \in \mathcal{V}_1$ describes the kinematic hardening behaviour, while the kinematic internal variable $\boldsymbol{\alpha}_2 \in \mathcal{V}_2$ describes the isotropic hardening/softening behaviour. The free energy functional $\mathit{\Phi} : \mathcal{D} \times \mathcal{V}_1 \times \mathcal{V}_2 \to \bar{\Re}$ (where $\bar{\Re} = \{-\infty \cup \Re \cup +\infty\}$) is expressed in decoupled form $\mathit{\Phi}(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = \mathit{\Phi}^{el}_L(\boldsymbol{\varepsilon}^e) + \mathit{\Phi}^{in}_{NL}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = \mathit{\Phi}^{el}_L(\boldsymbol{\varepsilon}^e) + \mathit{\Phi}^{kin}_{NL}(\boldsymbol{\alpha}_1) + \mathit{\Phi}^{iso}_{NL}(\boldsymbol{\alpha}_2)$, where $\mathit{\Phi}^{el}_L(\boldsymbol{\varepsilon}^e)$ is convex in $\boldsymbol{\varepsilon}^e$, the kinematic hardening potential $\mathit{\Phi}^{kin}_{NL}(\boldsymbol{\alpha}_1)$ is convex in $\boldsymbol{\alpha}_1$ and the isotropic hardening (softening) potential $\mathit{\Phi}^{iso}_{NL}(\boldsymbol{\alpha}_2)$ is convex (concave) in $\boldsymbol{\alpha}_2$. The nonlocal kinematic variables $\boldsymbol{\xi}_1 \in \mathcal{V}'_1$ and $\boldsymbol{\xi}_2 \in \mathcal{V}'_2$ are expressed as $\boldsymbol{\xi}_1(\mathbf{x}) = (\mathbf{R}\boldsymbol{\alpha}_1)(\mathbf{x})$ and $\boldsymbol{\xi}_2(\mathbf{x}) = (\mathbf{R}\boldsymbol{\alpha}_2)(\mathbf{x})$, where $\mathbf{R} : \mathcal{V}_i \to \mathcal{V}'_i$ is a suitable linear regularization operator. The nonlocal kinematic field $\boldsymbol{\xi}$ is determined as a weighted average of the local kinematic field $\boldsymbol{\alpha}$ by adopting the expression $\boldsymbol{\xi}(\mathbf{x}) = (\mathbf{R}\boldsymbol{\alpha})(\mathbf{x}) = \frac{1}{\mathit{\Omega}_r(\mathbf{x})} \int_{\mathit{\Omega}} g_{\mathbf{x}}(\mathbf{y}) \boldsymbol{\alpha}(\mathbf{y}) \mathrm{d}\, \mathit{\Omega}(\mathbf{y})$, where $g_{\mathbf{x}}(\mathbf{y}) = g(\|\mathbf{x} - \mathbf{y}\|)$ is a non-negative weighting function depending only on the distance $r = \|\mathbf{x} - \mathbf{y}\|$ between the source point $\mathbf{y} \in \mathit{\Omega}$ and the receiver point $\mathbf{x} \in \mathit{\Omega}$ and monotonically decreasing for $r \geq 0$, while $\mathit{\Omega}_r(\mathbf{x}) = \int_{\mathit{\Omega}} g_{\mathbf{x}}(\mathbf{y}) \mathrm{d}\, \mathit{\Omega}(\mathbf{y})$ is the reference volume. The static variables $\boldsymbol{\chi}_1 \in \mathcal{W}_1$ and $\boldsymbol{\chi}_2 \in \mathcal{W}_2$ are expressed as $\boldsymbol{\chi}_1 = \mathrm{d}_{\boldsymbol{\xi}_1} \mathit{\Phi}_{NL}(\boldsymbol{\xi}_1)$ and $\boldsymbol{\chi}_2 = \mathrm{d}_{\boldsymbol{\xi}_2} \mathit{\Phi}_{NL}(\boldsymbol{\xi}_2)$. For a linear behaviour it is $\boldsymbol{\chi}_1 = h_1 \boldsymbol{\xi}_1$ and $\boldsymbol{\chi}_2 = h_2 \boldsymbol{\xi}_2$, where h_1 is a positive kinematic hardening modulus and h_2 is an isotropic hardening/softening modulus. We also consider the dual average operator $\mathbf{R}' : \mathcal{W}_i \to \mathcal{W}'_i$ defined as the dual operator of $\mathbf{R}$, so that $\langle \boldsymbol{\chi}, \mathbf{R}\boldsymbol{\alpha} \rangle = \langle \mathbf{R}'\boldsymbol{\chi}, \boldsymbol{\alpha} \rangle, \ \forall \boldsymbol{\alpha} \in \mathcal{V}, \forall \boldsymbol{\chi} \in \mathcal{W}$. The symbol $\langle \bullet, \bullet \rangle$ herein denotes the inner product in the dual spaces. The dual average operator may be expressed as $\mathbf{X}(\mathbf{x}) = (\mathbf{R}'\boldsymbol{\chi})(\mathbf{x}) = \int_\Omega \frac{1}{\Omega_r(\mathbf{y})} g_\mathbf{x}(\mathbf{y})\boldsymbol{\chi}(\mathbf{y})\mathrm{d}\,\Omega(\mathbf{y})$. Consequently the nonlocal static internal variables $\mathbf{X}_1 \in \mathcal{W}_1'$ and $\mathbf{X}_2 \in \mathcal{W}_2'$ are determined as $\mathbf{X}_1(\mathbf{x}) = (\mathbf{R}'\boldsymbol{\chi}_1)(\mathbf{x})$ and $\mathbf{X}_2(\mathbf{x}) = (\mathbf{R}'\boldsymbol{\chi}_2)(\mathbf{x})$, and they represent the dual variables of the (local) kinematic internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. For the assumptions made above, it also results $\langle \boldsymbol{\chi}, \boldsymbol{\xi} \rangle = \langle \boldsymbol{\chi}, \mathbf{R}\boldsymbol{\alpha} \rangle = \langle \mathbf{R}'\boldsymbol{\chi}, \boldsymbol{\alpha} \rangle = \langle \mathbf{X}, \boldsymbol{\alpha} \rangle$. Consequently, the elastic energy and the potential functionals accounting for inelastic phenomena are expressed as $\Phi_L^{el}(\boldsymbol{\varepsilon}^e) = \frac{1}{2}\langle \mathbf{E}\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^e \rangle$, $\Phi_{NL}^{kin}(\boldsymbol{\alpha}_1) = \frac{1}{2}\langle h_1\boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle = \frac{1}{2}\langle h_1\mathbf{R}\boldsymbol{\alpha}_1, \mathbf{R}\boldsymbol{\alpha}_1 \rangle$ and $\Phi_{NL}^{iso}(\boldsymbol{\alpha}_2) = \frac{1}{2}\langle h_2\boldsymbol{\xi}_2, \boldsymbol{\xi}_2 \rangle = \frac{1}{2}\langle h_2\mathbf{R}\boldsymbol{\alpha}_2, \mathbf{R}\boldsymbol{\alpha}_2 \rangle$, where $\mathbf{E}$ is the elastic modulus. The constitutive relations are formulated as

$$(\boldsymbol{\sigma}, \mathbf{X}_1, \mathbf{X}_2) = \mathrm{d}\,\Phi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \iff \begin{cases} \boldsymbol{\sigma} = \mathrm{d}_{\boldsymbol{\varepsilon}^e}\Phi_L^{el}(\boldsymbol{\varepsilon}^e) = \mathbf{E}\boldsymbol{\varepsilon}^e, \\ \mathbf{X}_1 = \mathrm{d}_{\boldsymbol{\alpha}_1}\Phi_{NL}^{kin}(\boldsymbol{\xi}_1(\boldsymbol{\alpha}_1)) = \mathbf{R}'\mathrm{d}\,\Phi_{NL}^{kin}(\boldsymbol{\xi}_1) = \mathbf{R}'\boldsymbol{\chi}_1, \\ \mathbf{X}_2 = \mathrm{d}_{\boldsymbol{\alpha}_2}\Phi_{NL}^{iso}(\boldsymbol{\xi}_2(\boldsymbol{\alpha}_2)) = \mathbf{R}'\mathrm{d}\,\Phi_{NL}^{iso}(\boldsymbol{\xi}_2) = \mathbf{R}'\boldsymbol{\chi}_2, \end{cases} \tag{1}$$

and they are specialized as $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}^e$, $\mathbf{X}_1 = \mathbf{R}'\boldsymbol{\chi}_1 = \mathbf{R}'h_1\boldsymbol{\xi}_1(\boldsymbol{\alpha}_1) = (\mathbf{R}'h_1\mathbf{R})\boldsymbol{\alpha}_1$ and $\mathbf{X}_2 = \mathbf{R}'\boldsymbol{\chi}_2 = \mathbf{R}'h_2\boldsymbol{\xi}_2(\boldsymbol{\alpha}_2) = (\mathbf{R}'h_2\mathbf{R})\boldsymbol{\alpha}_2$. The elastic domain $\boldsymbol{C} = \{(\boldsymbol{\sigma}, \mathbf{X}_1, \mathbf{X}_2) \in \mathcal{S} \times \mathcal{W}_1' \times \mathcal{W}_2' : F(\boldsymbol{\sigma}, \mathbf{X}_1, \mathbf{X}_2) \le 0\}$ is defined by a convex yield function $F(\boldsymbol{\sigma}, \mathbf{X}_1, \mathbf{X}_2) = f(\boldsymbol{\sigma}, \mathbf{X}_1) - (\sigma_y + \mathbf{X}_2) = f(\boldsymbol{\sigma}, \mathbf{X}_1) - \mathbf{X}_2 - \sigma_y \le 0$, where f is a convex function and σ_y is a material parameter which characterizes the initial yield stress. The evolutive laws for the proposed model of nonlocal plasticity may be formulated for smooth problems as

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \dot\gamma\mathrm{d}_{\boldsymbol{\sigma}}F(\boldsymbol{\sigma}, \mathbf{X}_1, \mathbf{X}_2) = \dot\gamma\mathrm{d}_{\boldsymbol{\sigma}}f(\boldsymbol{\sigma}, \mathbf{X}_1), \\ -\dot{\boldsymbol{\alpha}}_1 = \dot\gamma\mathrm{d}_{\mathbf{X}_1}F(\boldsymbol{\sigma}, \mathbf{X}_1, \mathbf{X}_2) = \dot\gamma\mathrm{d}_{\mathbf{X}_1}f(\boldsymbol{\sigma}, \mathbf{X}_1), \\ -\dot{\boldsymbol{\alpha}}_2 = \dot\gamma\mathrm{d}_{\mathbf{X}_2}F(\boldsymbol{\sigma}, \mathbf{X}_1, \mathbf{X}_2) = -\dot\gamma, \end{cases} \tag{2}$$

with the complementarity conditions $f(\boldsymbol{\sigma}, \mathbf{X}_1) - \mathbf{X}_2 - \sigma_y \le 0,\ \dot\gamma \ge 0,\ \dot\gamma[f(\boldsymbol{\sigma}, \mathbf{X}_1) - \mathbf{X}_2 - \sigma_y] = 0$. The evolutive laws for the regularized nonlocal kinematic variables are therefore expressed as

$$\dot{\boldsymbol{\xi}}_1 = \mathbf{R}\dot{\boldsymbol{\alpha}}_1 = -\mathbf{R}[\dot\gamma\mathrm{d}_{\mathbf{X}_1}f(\boldsymbol{\sigma}, \mathbf{X}_1)], \qquad \dot{\boldsymbol{\xi}}_2 = \mathbf{R}\dot{\boldsymbol{\alpha}}_2 = \mathbf{R}\dot\gamma. \tag{3}$$

2. NONLOCAL MAXIMUM PLASTIC DISSIPATION

The flow laws of the proposed nonlocal plasticity model, together with the complementarity conditions, are the necessary and sufficient conditions for a nonlocal formulation of maximum plastic dissipation

$$\mathcal{D}^p(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}_1, -\dot{\boldsymbol{\alpha}}_2) = \sup_{(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) \in \boldsymbol{C}} \{\langle \bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}}^p \rangle - \langle \bar{\mathbf{X}}_1, \dot{\boldsymbol{\alpha}}_1 \rangle - \langle \bar{\mathbf{X}}_2, \dot{\boldsymbol{\alpha}}_2 \rangle\}. \tag{4}$$

By fulfilling a theorem of nonlocal maximum plastic dissipation the proposed constitutive formulation of nonlocal plasticity is equipped with a sound variational basis which provides the theoretical support for future developments.

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