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COUPLING DISCONTINUOUS GALERKIN METHODS AND RETARDED POTENTIALS FOR TIME DEPENDENT WAVE PROPAGATION PROBLEMS

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ABSTRACT

Discontinuous Galerkin (DG) methods have recently gained a great attention for the resolution of time dependent wave propagation problems. These methods benefit in particular from a great flexibility in terms of $h-p$ adaptivity. Moreover, they can be applied for solving PDE's such as linearized Euler equations (being aeroacoustics one of the applications we have in mind) where the presence of convection terms makes difficult the application of finite elements, for example.

On the other hand, many problems related with wave propagation are posed on unbounded domains. This raises the question of bounding artificially the computational domain. The increasing computational power together with the progress of rapid algorithms like the fast multipole method make now possible, at least when the exterior domain is homogeneous, the use of exact or transparent boundary conditions. Such condition relies on an explicit representation of the solution on the exterior domain Ω_e from its traces on the boundary Γ. In 3D, these formulas are known as retarded potential representations (RP).

That is why one is naturally lead to investigate the question of coupling these two techniques. One of the major difficulties is to guarantee a priori the stability of the resulting scheme. The method we shall present on this talk is based on a combination and adaptation of the ideas presented in [1,2]. We shall present this method on the model problem of the acoustic wave equation (but it can be generalized to other Friedrichs' systems). The problem is formulated as a system between the PDE's for the pressure p and the velocity $\boldsymbol{v}$ in the interior domain Ω_i and $\varphi \equiv p_{|\Gamma}$ and $\psi \equiv (\mathbf{v}\cdot\mathbf{n})_{|\mathbf{\Gamma}}$ on its boundary Γ, namely, the traces of the solution on the interface. More precisely, we couple the interior PDE's for p_i and $\boldsymbol{v}_i$

$$\left| \begin{array}{rcrcll} \rho \dfrac{\partial p_i}{\partial t} & - & \mathrm{div}\boldsymbol{v}_i & = & f, & \text{in } \Omega_i, \\ \boldsymbol{A}\dfrac{\partial \boldsymbol{v}_i}{\partial t} & - & \nabla p_i & = & 0, & \text{in } \Omega_i, \end{array} \right. \tag{1}$$

with the transparent condition written as

$$\left| \begin{array}{ccccccl} \dfrac{1}{2}\,\boldsymbol{v}_i \cdot \boldsymbol{n} & + & \mathcal{Y}_\Gamma \varphi & + & \mathcal{W}^*_\Gamma \psi & = & 0, \quad \text{on } \Gamma, \\ \dfrac{1}{2}\,p_i & + & \mathcal{Z}_\Gamma \psi & - & \mathcal{W}_\Gamma \varphi & = & 0, \quad \text{on } \Gamma, \end{array} \right. \tag{2}$$

where

$$\begin{bmatrix} \mathcal{Z}_\Gamma & \mathcal{W}^*_\Gamma \\ -\mathcal{W}_\Gamma & \mathcal{Y}_\Gamma \end{bmatrix} \tag{3}$$

is the classical Calderón-Zygmund operator associated to the wave equation. This is a space-time integral operator acting on functions defined on $\Gamma \times \mathbb{R}^+$ whose expression is computed from the fundamental solution of the wave equation [3]. We can write a *DG type* variational formulation of (1) to obtain the following coupled problem

$$\left| \begin{array}{lllll} m(\dfrac{\partial \boldsymbol{u}_i}{\partial t}, \widetilde{\boldsymbol{u}}_i) \; + \; a_h(\boldsymbol{u}_i, \widetilde{\boldsymbol{u}}_i) \; - \; c(\boldsymbol{u}_\Gamma, \widetilde{\boldsymbol{u}}_i) & = & f(\widetilde{\boldsymbol{u}}_i), & & \forall\, \widetilde{\boldsymbol{u}}_i, \\ b_T(\boldsymbol{u}_\Gamma, \widetilde{\boldsymbol{u}}_\Gamma) \; + \; \displaystyle\int_0^T c(\widetilde{\boldsymbol{u}}_\Gamma, \boldsymbol{u}_i)\, \mathrm{d}t & = & 0, & & \forall\, \widetilde{\boldsymbol{u}}_\Gamma, \end{array} \right. \tag{4}$$

where $\boldsymbol{u}_i = (p_i, \boldsymbol{v}_i)$ and $\boldsymbol{u}_\Gamma = (\varphi, \psi)$. In this last system, $m(\cdot,\cdot)$ is the standard mass bilinear form, $a_h(\cdot,\cdot)$ is the skew symetric stiffness form associated to the DG approximation with central fluxes on a mesh $\mathcal{T}_h$ of Ω_i and $c(\cdot,\cdot)$ is the coupling bilinear form given by

$$c(\boldsymbol{u}_\Gamma, \boldsymbol{u}_i) \;=\; \frac{1}{2} \int_\Gamma [\, p_i\, \psi + \varphi\, \boldsymbol{v}_i \cdot \boldsymbol{n}\,]\; \mathrm{d}\gamma. \tag{5}$$

Finally, the most delicate step is the space-time discretization of (4) combining DG in space and finite differences in time for the first equation and a space-time Galerkin approach for the second. The resulting scheme is explicit at the interior of the domain, subject to a CFL condition, and implicit on the boundary through the inversion of a matrix with small bandwidth determined by the time step. The key point is to design the discretization procedure in order to guarantee a discrete energy identity that is consistent with the continuous one and yields stability under the CFL condition of the interior scheme (in other words, the stability condition is not affected by the coupling with the transparent boundary conditions).

This construction will be detailed during the conference together with the main theoretical results and computational issues. Numerical experiments on academic test cases will be presented.

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