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Natural convection/isothermal viscous incompressible flows by the velocity-pressure-vorticity formulation.

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ABSTRACT

2D viscous natural convection/isothermal incompressible flows are presented from the unsteady Boussinesq approximation and the Navier-Stokes equations in its velocity-pressure-vorticity formulation. The results are obtained using simple numerical procedures: After time discretization, a fixed point iterative process to solve the resulting nonlinear elliptic system is used in one case and a projection method in the other. In either case, the solution of uncoupled, symmetric linear elliptic problems is required for which efficient solvers exist regardless of the space discretization

Let $\Omega \subset R^N (N = 2, 3)$ be the region of the unsteady flow of a viscous incompressible thermal fluid, and Γ its boundary. This kind of flows is governed by the non-dimensional system, in $\Omega \times (0, T)$, $T > 0$,

$$\begin{cases} \mathbf{u}_t - \frac{1}{Re}\nabla^2 \mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} & (a) \\ \nabla \cdot \mathbf{u} = 0 & (b) \\ \theta_t - \frac{1}{RePr}\nabla^2\theta + \mathbf{u}\cdot\nabla\theta = 0 & (c) \end{cases} \tag{1}$$

known as the Boussinesq approximation in primitive variables if $\mathbf{f} = \frac{Ra}{PrRe^2}\,\theta\,\mathbf{e}$ where $\mathbf{u}$, p, and θ are the velocity, pressure, and temperature of the flow, and $\mathbf{e}$ is the unitary vector in the gravitational direction. The dimensionless parameters Re, Ra and Pr are the Reynolds, Rayleigh and Prandtl numbers respectively. If the flow does not depend on the temperature, the coupling with $(1c)$ is eliminated and $\mathbf{f}$ does not depend on θ, then the Navier-Stokes equations for isothermal flows is given by $(1a - b)$. The system must be supplemented with initial and boundary conditions in Ω and on Γ respectively.

Taking the curl in $(1a)$, the vorticity vector ω transport equation in $\Omega \times (0, T)$ reads

$$\begin{cases} \omega_t - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = \omega\cdot\nabla\mathbf{u} + \mathbf{f} & \text{(a)} \\ \omega = \nabla \times \mathbf{u}, & (b) \end{cases} \tag{2}$$

the new $\mathbf{f}$ is the curl of the old one. Taking the curl in $(2b)$ and using $(1b)$, from the identity $\nabla\times\nabla\times\mathbf{a} = -\nabla^2\mathbf{a} + \nabla(\nabla\cdot\mathbf{a})$, the following velocity Poisson equation is obtained

$$\nabla^2\mathbf{u} = -\nabla\times\omega \tag{3}$$

Hence, equations $(2a)$ and (3) coupled to $(1c)$ give the Boussinesq approximation in velocity-vorticity formulation. It can be easily verified that the vorticity, scalar, ω transport equation in $\Omega \times (0, T)$, $\Omega \subset R^2$, is given by

$$\omega_t - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = \frac{Ra}{PrRe^2}\frac{\partial\theta}{\partial x} \tag{4}$$

where, from the $2D$ restriction in $(2b)$,

$$\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{5}$$

and, from (3), the two Poisson equations for the velocity components are expressed as

$$\left\{ \nabla^2 u_1 = -\frac{\partial\omega}{\partial y} \qquad (a)\,, \quad \nabla^2 u_2 = \frac{\partial\omega}{\partial x} \qquad (b) \right. \tag{6}$$

Then, the vector Boussinesq approximation system $(2a)$ and (3) coupled to $(1c)$ leads to a scalar system of four equations in $2D$: one given by (4) and two by (6), both coupled to $(1c)$; the boundary condition for ω in (4) is obtained from that of $\mathbf{u} = (u_1, u_2)$ through (5).

After a second order discretization of the time derivatives appearing in the vorticity and temperature equations, a nonlinear system of elliptic equations of the following form has to be solved in Ω

$$\begin{cases} \nabla^2 u_1 = -\frac{\partial\omega}{\partial y} & \\ \nabla^2 u_2 = \frac{\partial\omega}{\partial x}, & \mathbf{u} = \mathbf{u}_{bc} \;\; on\; \Gamma \\ \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = \frac{Ra}{PrRe^2}\frac{\partial\theta}{\partial x} + f_\omega, & \omega = \omega_{bc} \;\; on\; \Gamma \\ \alpha\theta - \gamma\Delta\theta + \mathbf{u}\cdot\nabla\theta = f_\theta, & B\theta|_\Gamma = 0, \end{cases} \tag{7}$$

where $\alpha = \frac{3}{2\Delta t}$; f_ω and f_θ are the contributions of the known time levels. $\frac{1}{Re}$=ν= kinematic viscosity, and $\gamma = \frac{1}{PrRe^2}$; $\mathbf{u}_{bc}$ and ω_{bc} denote the corresponding boundary condition for $\mathbf{u}$ and ω, and $B\theta$ a boundary operator for θ that can involve Dirichlet, Neumann or mixed boundary conditions. This work involves isothermal flows, governed by the Navier-Stokes equations and natural convection flows under the Boussinesq approximation; then, for the latter case, $Re = 1$, [1].

Since the elliptic system (7) is nonlinear and of non-potential type, a fixed point iterative process is used to solve it being either an adaptation to natural convection of the one for mixed convection in the stream function-vorticity formulation, [2], or an extension to natural convection of the one for isothermal flows in velocity-vorticity formulation, [3]; whereas, for the velocity-pressure system (1) a projection method based on an operator splitting is applied.

The numerical experiments take place in rectangular cavities, then $\Omega = (0, a) \times (0, b)$ with a, $b > 0$. Then, by viscosity and since for natural convection all the walls of the cavity are solid and fixed, the boundary condition for $\mathbf{u}$ is $\mathbf{0}$ everywhere on Γ whereas that for θ is given by

$$\left\{ \theta = 1 \; on \; \Gamma|_{x=0}, \; \theta = 0 \; on \; \Gamma|_{x=a}; \quad \frac{\partial\theta}{\partial n} = 0 \; on \; \Gamma|_{y=0,b}, \right.$$

hence the horizontal walls are insulated and the left wall is the hot one; it is assumed that the cavity is filled with air, then $Pr = 0.72$. For isothermal flows, the well known driven cavity problem is considered: $\mathbf{u}(x, b) = (1, 0)$ on the moving wall $y = b$ and $\mathbf{0}$ elsewhere. The initial conditions are given by $\mathbf{u}(\mathbf{x}, 0) = (0, 0)$ and $\omega(\mathbf{x}, 0) = 0 = \theta(\mathbf{x}, 0)$ in Ω. Results converging to the asymptotic steady state are reported at moderate Rayleigh and Reynolds numbers.

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