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A time simulation based approach to numerical bifurcation analysis of fluid problems

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ABSTRACT

Bifurcation analysis is a technique to study stable and unstable steady states and periodic solutions of parameter-dependent dynamical systems. In a bifurcation study, branches of solutions are computed whereby varying a parameter of the system, and parameter values where the linear stability of the system changes, are determined and classified.

For small ODE systems and low-dimensional maps, numerical bifurcation analysis has been very successful. Several public-domain codes are available. AUTO [3] and Matcont [4] are two popular examples. Codes for PDEs, such as LOCA [6] and ENTWIFE [1], are often based on extensions of the methods used for ODEs. They are suitable for PDEs that can be discretised using a method-of-lines approach. These codes differ from those for small ODE systems in the linear algebra techniques used to solve the linear systems and the eigenvalue problems.

There is however another class of methods: time simulation-based methods. These methods are particularly interesting when a time simulation code is already available or when a method-of-lines discretisation is not adequate (as is sometimes the case in computational fluid dynamics). Moreover, these techniques can also be used with other (i.e., non-PDE) time simulation codes, e.g., lattice Boltzmann models [9]. Time simulation-based bifurcation analysis is a very versatile approach which can be used to compute many different kinds of solutions: steady states, periodic solutions of autonomous and non-autonomous problems, travelling waves, spiral waves, etc. It is also a natural extension of the way engineers use time simulation to study a system.

Time simulation-based methods are extensions of methods for bifurcation analysis of maps. The map in this case is related to the time simulator. The different solution types mentioned before have one thing in common: one must compute a fixed point of a (high-dimensional) map subject to a small number of algebraic constraints. This requires the solution of a large nonlinear system. For large-scale systems, it is essential to exploit properties of the problem to compute a solution efficiently. In the

traditional framework, the sparsity of the Jacobian matrices is exploited. However, in time simulation based methods, the Jacobian matrix is a dense matrix and only the spectrum can be exploited. Several methods have been proposed in the literature to tackle this problem, including Newton-Krylov methods [e.g., [7]), the Recursive Projection Method [8] and variants of Broyden's method [4]. We developed the Newton-Picard method [5] which works well for problems for which the time map has a spectrum that is clustered about 0.

The Newton-Picard method combines time simulation used as a fixed point iteration (sometimes called Picard iteration) with the Newton-Raphson method. The basic idea is to use Newton's method only in the directions in which the time integrations diverge or converge too slowly. The end result is an efficient numerical method which combines the computation of a steady-state or periodic solution with the stability analysis of this solution. Moreover, the particular structure of the method makes the method particularly interesting for numerical bifurcation analysis.

We have applied the method successfully to fluid problems such as 2D flow behind a cylinder and axisymmetric opposed jet flow (joint work with C. E. Frouzakis, A. Ciani, W. Kreutner and K. Boulouchos) and are currently also experimenting with ocean and atmosphere models (joint work with H. Dijkstra and coworkers). In the cylinder problem we determined the first Hopf bifurcation and computed the resulting branch of periodic solutions up to a Reynolds number about 300. In the opposed jet model, a symmetry-breaking bifurcation of steady state solutions was located, and both of the resulting stable asymmetric solution branches and the unstable symmetric solution branch were computed. In both cases, the velocity field was used to represent the state. The pressure was not included in the state. Even though the velocity unknowns themselves are not independent of each other, no further measures had to be taken to take the DAE character of the Navier-Stokes equations into account. For both test problems, the method worked reliably and the results corresponded nicely to those in the literature (cylinder problem) or from experiments (opposed jet flow).

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