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SOLID WITH AN IMMERSED THIN BEAM

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ABSTRACT

Problem description We consider the elasticity problem of three domains with different compliances as shown in the left picture of Figure 1. The domain ω in the middle is thin and has a significantly

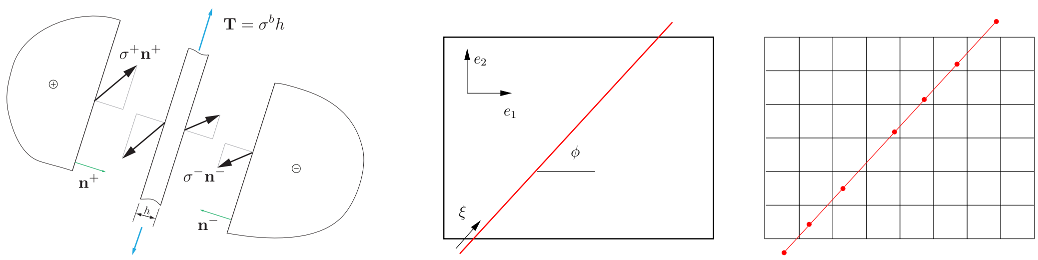

Figure 1: A stiff solid with thickness h (center) embedded between two compliant solids and its discretization with finite elements.

different compliance than the two adjacent domains Ω^+ and Ω^-, so that it can be approximated as a beam. Hence, the overall problem can be considered as a beam ω embedded in a two dimensional solid $\Omega = \Omega^+ \cup \Omega^-$ (Fig. 1, middle). The governing equations for the solid Ω are

$$-\text{div}\,\sigma(\boldsymbol{u}) = \boldsymbol{f}, \quad \sigma(\boldsymbol{u}) = C(\epsilon(\boldsymbol{u})), \quad \epsilon(\boldsymbol{u}) = \frac{1}{2}\left(\text{grad}\,\boldsymbol{u} + \text{grad}\,\boldsymbol{u}^{\mathrm{T}}\right), \tag{1}$$

where σ is the stress tensor, $\boldsymbol{f}$ the external load vector, ϵ the strain tensor, $\boldsymbol{u}$ the displacement vector, and $C(\cdot)$ realizes the usual linear stress-strain relationship. For brevity, it is assumed that the beam has zero bending stiffness, which results in the governing equations for the beam:

$$-T'(w) = q, \quad T(w) = D(\chi(w)), \quad \chi(w) = w', \tag{2}$$

where T is the axial force, χ the axial strain, w the axial displacements, and $\cdot'$ denotes differentiation in axial direction. At the beam-solid interface the displacement continuity conditions are

$$[\boldsymbol{u} \cdot \boldsymbol{n}] = 0, \qquad \boldsymbol{u}^i \cdot \boldsymbol{t} = w, \ i = +, -, \tag{3}$$

where $[\cdot]$ denotes the jump over the beam. The tangential and normal traction continuity equations are

$$[\sigma_n(\boldsymbol{u})] = 0, \qquad [\sigma_t(\boldsymbol{u})] = q, \tag{4}$$

where σ_n and σ_t is the normal and tangential component of the surface traction $\sigma\boldsymbol{n}$, respectively. Since bending effects are neglected, the normal tractions have to be continuous across the solid-beam interface. Physical considerations and dimensional analysis explain the fact that the jump in the tangential traction is equal to the change of the beam axial force, resulting in a at first glance quite unusual coupling of the solid Neumann data with the beam source term.

Weak formulation We aim to minimize the following energy functional (neglecting any boundary conditions):

$$\begin{aligned}
\Pi(\boldsymbol{u}, w) &= \Pi_\Omega(\boldsymbol{u}, w) + \Pi_\Gamma(\boldsymbol{u}, w), \\
\Pi_\Omega(\boldsymbol{u}, w) &= \tfrac{1}{2}(\sigma(\boldsymbol{u}), \epsilon(\boldsymbol{u}))_\Omega - (\boldsymbol{f}, \boldsymbol{u})_\Omega + \tfrac{1}{2}\|\lambda[\boldsymbol{u} \cdot \boldsymbol{n}]\|_\Gamma^2 - (\sigma_n(\boldsymbol{u}), [\boldsymbol{u} \cdot \boldsymbol{n}])_\Gamma \\
&\quad + \tfrac{1}{2}\textstyle\sum_i \|\lambda(\boldsymbol{u}^i \cdot \boldsymbol{t} - w)\|_\Gamma^2 - \textstyle\sum_i (\sigma_t(\boldsymbol{u}^i), \boldsymbol{u}^i \cdot \boldsymbol{t} - w)_\Gamma, \\
\Pi_\Gamma(\boldsymbol{u}, w) &= \tfrac{1}{2}(T(w), \chi(w))_\Gamma - ([\sigma_t(\boldsymbol{u})], w)_\Gamma.
\end{aligned}$$

In particular, the displacement coupling condition (3) is enforced by penalization with a suitable parameter λ, where the terms involving σ_n and σ_t in the definition of Π_Ω have to be added for consistency. The traction coupling condition (4) is realized by replacing the beam source term q by $[\sigma_t(\boldsymbol{u})]$.

Discretization and loose coupling procedure For the discretization, we use a triangulation of Ω which does not need to take into account the presence of the beam, see Figure 1 (right). For the solution of the fully coupled system, we use the following loose coupling approach:

for $k = 0, \ldots$ **do**
1. Given w^k, calculate $\boldsymbol{u}^{k+1}$ by minimizing $\Pi_\Omega(\boldsymbol{u}, w^k)$.
2. Given $\boldsymbol{u}^{k+1}$, calculate w^{k+1} by minimizing $\Pi_\Gamma(\boldsymbol{u}^{k+1}, w)$.

end for

The first step requires to solve a discrete linear elasticity problem with given Dirichlet conditions on an internal interface. With the employed Nitsche type penalization, this results in a modified version of the method proposed in [1]. The second step just involves the solution of the discrete beam equation with given source term. In both steps, special care has to be taken to evaluate a discrete function from one mesh with respect to the other mesh.

Outlook The proposed method is to be seen as an intermediate step towards the simulation of fluid-structure interaction problems, where the structure can be modelled using a reduced dimension. Here, the analogous approach is especially attractive if the motion of the structure has to be taken into account.

REFERENCES

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