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ENERGY BUDGET IN A SUB-GRID SCALE FINITE ELEMENT MODEL FOR TURBULENT INCOMPRESSIBLE FLOWS

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ABSTRACT

Let us consider the incompressible Navier-Stokes equations

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u} = 0$$

on a domain $\Omega \subset \mathbb{R}^d$, during a time interval $[0,T]$, with boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ and an initial condition $\boldsymbol{u} = \boldsymbol{u}_0$ at $t = 0$. In this work we describe a finite element method to approximate the solution to this problem based on a variational scale decomposition (see [1]) into the component of the velocity and the pressure in the corresponding finite element space and their subscales, an approximation for which will be proposed. To simplify the exposition, we will consider only a velocity subscale.

Let $V = (H_0^1(\Omega)^d)$, $Q = L_0^2(\Omega)$. The variational form of the problem is

$$(\partial_t\boldsymbol{u},\boldsymbol{v}) + \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + \langle\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}\rangle - (p,\nabla\cdot\boldsymbol{v}) + (q,\nabla\cdot\boldsymbol{u}) = \langle\boldsymbol{f},\boldsymbol{v}\rangle, \quad \forall\,[\boldsymbol{v},q]\in V\times Q \tag{1}$$

which holds a.e. in $[0,T]$. Here, (f,g) denotes the L^2-inner product of functions f and g and $\langle f,g\rangle = \int_\Omega fg$, whenever this integral makes sense. Let now $V_h \subset V$ and $Q_h \subset Q$ be finite element spaces for V and Q, respectively. Let us consider the approximations $\boldsymbol{u}(\cdot,t) \approx \boldsymbol{u}_h(\cdot,t) + \boldsymbol{u}'(\cdot,t)$, $p(\cdot,t)\approx p_h(\cdot,t)$, with $\boldsymbol{u}_h(\cdot,t)\in V_h$, $p_h(\cdot,t)\in Q_h$ and $\boldsymbol{u}'(\cdot,t)$ in a space of velocity subscales V' to be defined. If we call $\mathcal{L}_u\boldsymbol{v} := \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \nu\Delta\boldsymbol{v}$ and $\mathcal{L}_u^*\boldsymbol{v} := -\boldsymbol{u}\cdot\nabla\boldsymbol{v} - \nu\Delta\boldsymbol{v}$, integrating some terms by parts and using the continuity of the normal stresses across interelement boundaries, the approximate version of (1) is

$$\begin{aligned}&(\partial_t\boldsymbol{u}_h,\boldsymbol{v}_h) + \nu(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}_h) + \langle\boldsymbol{u}\cdot\nabla\boldsymbol{u}_h,\boldsymbol{v}_h\rangle - (p_h,\nabla\cdot\boldsymbol{v}_h) + (q_h,\nabla\cdot\boldsymbol{u}_h)\\ &\quad + (\partial_t\boldsymbol{u}',\boldsymbol{v}_h) + \sum_K\langle\boldsymbol{u}',\mathcal{L}_u^*\boldsymbol{v}_h - \nabla q_h\rangle_K = \langle\boldsymbol{f},\boldsymbol{v}_h\rangle, \quad \forall[\boldsymbol{v}_h,q_h]\in V_h\times Q_h\end{aligned} \tag{2}$$

$$(\partial_t\boldsymbol{u}',\boldsymbol{v}') + \sum_K(\langle\mathcal{L}_u\boldsymbol{u}',\boldsymbol{v}'\rangle_K + \langle\partial_t\boldsymbol{u}_h + \mathcal{L}_u\boldsymbol{u}_h + \nabla p_h,\boldsymbol{v}'\rangle_K) = \langle\boldsymbol{f},\boldsymbol{v}'\rangle, \quad \forall\boldsymbol{v}'\in V' \tag{3}$$

where $\sum_K$ stands for the summation over all the elements K of the finite element partition. The next step is to make the approximation $\mathcal{L}_u\boldsymbol{u}' = \tau^{-1}\boldsymbol{u}'$ for a certain parameter τ. This approximation can be motivated from a Fourier argument, as explained in [2]. In this case, (3) implies

$$\partial_t\boldsymbol{u}' + \tau^{-1}\boldsymbol{u}' = -P_h'(\partial_t\boldsymbol{u}_h + \mathcal{L}_u\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) \tag{4}$$

where P_h' is the projection onto V' (not yet defined) with respect to $\sum_K \langle \cdot, \cdot \rangle_K$.

Let R be a region formed by a patch of elements, and let $\boldsymbol{t}_R$ be the consistent finite element flux on ∂R. Denoting by a subscript R integrals restricted to R, the balance of kinetic energy in R is obtained taking $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (2) and testing (4) with $\boldsymbol{u}'$, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_h\|_R^2 + \nu\|\nabla\boldsymbol{u}_h\|_R^2 + (\partial_t\boldsymbol{u}', \boldsymbol{u}_h)_R + \sum_{K\subset R}\langle\boldsymbol{u}', P_h'(\mathcal{L}_u^*\boldsymbol{u}_h - \nabla p_h)\rangle_K = \langle\boldsymbol{f}, \boldsymbol{u}_h\rangle_R + \langle\boldsymbol{t}_R, \boldsymbol{u}_h\rangle_{\partial R}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}'\|_R^2 + \tau^{-1}\|\boldsymbol{u}'\|_R^2 + (\partial_t\boldsymbol{u}_h, \boldsymbol{u}')_R + \sum_{K\subset R}\langle\boldsymbol{u}', P_h'(\mathcal{L}_u\boldsymbol{u}_h + \nabla p_h)\rangle_K = \langle\boldsymbol{f}, \boldsymbol{u}'\rangle_R$$

From these expressions, which clearly display the transfer of energy between finite element and subgrid scales, way may draw the following conclusion:

1 *The finite element and the subgrid scales do not lead to a proper scale separation in the kinetic energy balance unless $V' = V_h^\perp$, that is, the subscales are orthogonal to the finite element space.*

This is the possibility advocated in [2]. Suppose now that this orthogonality holds and that $\boldsymbol{f} \in V_h$. From (4) it follows that

$$\langle\boldsymbol{f}, \boldsymbol{u}_h\rangle_R + \langle\boldsymbol{t}_R, \boldsymbol{u}_h\rangle_{\partial R} = \frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_h\|_R^2 + \nu\|\nabla\boldsymbol{u}_h\|_R^2 + \underbrace{\sum_{K\subset R}\tau\langle P_h'(\mathcal{L}_u\boldsymbol{u}_h + \nabla p_h), P_h'(-\mathcal{L}_u^*\boldsymbol{u}_h + \nabla p_h)\rangle_K}_{(I)} + \underbrace{\sum_{K\subset R}\tau\langle\partial_t\boldsymbol{u}', P_h'(-\mathcal{L}_u^*\boldsymbol{u}_h + \nabla p_h)\rangle_K}_{(II)}$$

For the two terms coming from the subscales, the following can be said:

2 *Assuming fully developed turbulent flow, term (I) can be shown to behave as the molecular dissipation of the physical subscales (see [3]).*

3 *Term (II) can be both positive and negative. It is the only term able to model backscatter. Note that it requires dynamic subscales, a concept fully developed in [4].*

The purpose of the work to be presented is to elaborate on conclusions 1, 2 and 3.

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