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THE CHARACTERISTIC / INERTIAL GALERKIN METHOD APPLIED TO SHALLOW WATER FLOWS

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ABSTRACT

In this work we discuss the instability encountered in flows dominated by the Coriolis force and propose a numerical strategy to cope with this problem. It is an application of the ideas first introduced in [1] to flows in shallow waters where the rotating terms are important, a typical situation in oceanographic flows.

In the first part, we obtain the expression for the acceleration of a fluid particle when it is referred to a rotating frame of reference. This development is absolutely standard and basic, but will serve to motivate the numerical method introduced in the third part of the presentation.

Once the shallow water equations are written in a rotating reference system and discretized in time, a simple energy estimate is obtained. This is a standard development, but serves to present the instability encountered when rotation dominates at the same point and with the same arguments as the instabilities due to convection dominated flows.

As an extension of the well known Characteristic Galerkin Method, the Inertial Galerkin Method is then introduced as a remedy to deal with the instabilities due to convection and rotation dominated flows at the same time.

If $\boldsymbol{e}_i(t)$ is the i-th component of the rotating reference system (expressed in terms on an inertial one) and $u_i(\boldsymbol{x}(\boldsymbol{X},t),t)$ is the i-th velocity component of the particle that at time $t = 0$ is located at the spatial point $\boldsymbol{X}$, being $\boldsymbol{x}(\boldsymbol{X},t)$ the trajectory of this particle, the essential idea of the method is to discretize the total time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u_i(\boldsymbol{x}(\boldsymbol{X},t),t)\boldsymbol{e}_i(t)\right) \tag{1}$$

In the case the basis does not depend on time, this idea is the starting point of the characteristic Galerkin or Lagrange-Galerkin method. In the case of shallow water flows, the derivative (1) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u_i\boldsymbol{e}_i\right) = \left(\frac{\partial u_1}{\partial t} + u_j\frac{\partial u_1}{\partial x_j} - fu_2\right)\boldsymbol{e}_1 + \left(\frac{\partial u_2}{\partial t} + u_j\frac{\partial u_2}{\partial x_j} + fu_1\right)\boldsymbol{e}_2 \tag{2}$$

where u_i is the i-th depth averaged velocity and f is the Coriolis parameter.

Even though (1) and (2) are two expressions of the same temporal derivative, the time discretization of (1) or of the temporal derivative in (2) lead to different schemes. If we denote with a superscript the time step level of a typical finite difference time discretization, a two-step approximation of (1) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u_i(\boldsymbol{x}(\boldsymbol{X},t),t)\boldsymbol{e}_i(t)\right)\bigg|_{t^{n+\theta}} \approx \frac{1}{\delta t}\big(u_i(\boldsymbol{x}(\boldsymbol{X},t^{n+1}),t^{n+1})\boldsymbol{e}_i(t^{n+1}) - u_i(\boldsymbol{x}(\boldsymbol{X},t^n),t^n)\boldsymbol{e}_i(t^n)\big)$$

with $0 \leq \theta \leq 1$ and δt being the time step size. Once arrived to this expression, rather than integrating the equations along the characteristics and referring the time levels to a same inertial basis, a Taylor expansion is performed to obtain a problem posed in the current reference and the current spatial points. The outcome is the appearance of some stabilizing terms that will enhance stability.

We present some numerical examples showing the importance of the Coriolis forces in some problems of interest, as well as the local instabilities they may lead to if they are not properly accounted for in the numerical discretization of the equations.

REFERENCES

[1] R. Codina. “Numerical solution of the incompressible Navier-Stokes equations with Coriolis forces based on the discretization of the total time derivative”. *Journal of Computational Physics*, Vol. **148**, 467–496 (1999).