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MIXED FINITE APPROXIMATION OF THE MATERIAL FAILURE PROCCESS WITH CONTINUUM DAMAGE MODELS: SHEAR FAILURE MATERIAL MODE

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ABSTRACT

This paper uses a general non linear variational formulation and approximation of the material failure process within the framework of continuum damage mechanics and Finite Elements. The formulation is based on the Fraeijs de Veubeke functional (FV) for continuum elastic solids, which considers as independent variables, displacements **u**, stresses **σ**, strains **ε** and tractions **t**.

$$\Pi_{\mathrm{FV}}\left(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{t}\right) \equiv \int_{\Omega}\left[\boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon}^{u}-\boldsymbol{\varepsilon}\right)+W(\boldsymbol{\varepsilon})-\mathbf{b}^{*}\cdot\mathbf{u}\right]d\Omega-\int_{\Gamma_{\sigma}}\mathbf{t}^{*}\cdot\mathbf{u}\,d\Gamma+\int_{\Gamma_{u}}\mathbf{t}\cdot(\mathbf{u}-\mathbf{u}^{*})\,d\Gamma \quad (1)$$

where W(ε) is the strain energy density, **b** is the body force vector and $\mathbf{u}^*$ and $\mathbf{t}^*$ are, respectively, the prescribed displacements and tractions on boundary Γ.

It is shown that this formulation leads to a displacement (D) and two different mixed functionals, the Hellinger-Reissner functional (HR) with displacements and stresses as independent variables and the strain-displacement functional (SD) with displacements and strains as independent variables. These three functionals are given, respectively, by the equations:

$$\Pi_{\mathrm{D}}\left(\mathbf{u}\right) \equiv \int_{\Omega}\left[W(\boldsymbol{\varepsilon})-\mathbf{b}^{*}\cdot\mathbf{u}\right]d\Omega-\int_{\Gamma_{\sigma}}\mathbf{t}^{*}\cdot\mathbf{u}\,d\Gamma+\int_{\Gamma_{u}}\boldsymbol{\sigma}^{\boldsymbol{\varepsilon}^{u}}\cdot\boldsymbol{\nu}\cdot(\mathbf{u}-\mathbf{u}^{*})\,d\Gamma \quad (2)$$

$$\Pi_{\mathrm{HR}}\left(\mathbf{u},\boldsymbol{\sigma}\right) \equiv \int_{\Omega}\left[\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^{u}-\varphi(\boldsymbol{\sigma})-\mathbf{b}^{*}\cdot\mathbf{u}\right]d\Omega-\int_{\Gamma_{\sigma}}\mathbf{t}^{*}\cdot\mathbf{u}\,d\Gamma+\int_{\Gamma_{u}}\boldsymbol{\sigma}\cdot\boldsymbol{\nu}\cdot(\mathbf{u}-\mathbf{u}^{*})\,d\Gamma \quad (3)$$

$$\Pi_{\mathrm{DS}}\left(\mathbf{u},\boldsymbol{\varepsilon}\right) \equiv \int_{\Omega}\left[\boldsymbol{\sigma}^{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}^{u}-W(\boldsymbol{\varepsilon})-\mathbf{b}^{*}\cdot\mathbf{u}\right]d\Omega-\int_{\Gamma_{\sigma}}\mathbf{t}^{*}\cdot\mathbf{u}\,d\Gamma+\int_{\Gamma_{u}}\boldsymbol{\sigma}^{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\nu}\cdot(\mathbf{u}-\mathbf{u}^{*})\,d\Gamma \quad (4)$$

where φ(σ) is the complementary energy density.

In both, displacement or mixed finite element approximations of this problem, damage is smeared over the whole element using a continuum constitutive model; however, in a mixed formulation, secondary values such as strains or stresses are approximated independently from the primary variables, displacements, allowing for an improved

distribution of damage [1].

In this formulation, the nonlinear material behaviour is represented by a continuum damage models. To cover a range of materials of interest two families of damage models are considered: isotropic continuum and elasto-plastic, both equipped with softening. The main components of these models are [2]:

	Isotropic continuum damage model	Elastoplastic model with isotropic softening
Free energy	$\Psi(\boldsymbol{\varepsilon},r)=(1-d(r))\Psi_0;\ \ \Psi_0=\frac{1}{2}\boldsymbol{\varepsilon}:\mathbf{C}^e:\boldsymbol{\varepsilon}$ $d(r)=1-\frac{q}{r}$	$\Psi(\boldsymbol{\varepsilon}^e,r)=\frac{1}{2}\boldsymbol{\varepsilon}^e:\mathbf{C}^e:\boldsymbol{\varepsilon}^e+\Psi^p(r)$ $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^e+\boldsymbol{\varepsilon}^p$
Internal variables	$\dot{r}=\dot{\lambda},\ r\big\vert_{t=0}=r_0=\sigma_u/\sqrt{G}$	$\dot{r}=\dot{\lambda},\ r\big\vert_{t=0}=0$ $\dot{\boldsymbol{\varepsilon}}^p=\dot{\lambda}\xi,\ \xi=\partial_\sigma\varphi(\boldsymbol{\sigma},q)$
Constitutive equation	$\sigma=\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}=(1-d)\underbrace{\mathbf{C}^e:\boldsymbol{\varepsilon}}_{\bar{\boldsymbol{\sigma}}}=\frac{q}{r}\mathbf{C}^e:\boldsymbol{\varepsilon}$	$\sigma=\mathbf{C}^e:\boldsymbol{\varepsilon}^e=\mathbf{C}^e:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$
Damage criterion	$g(\boldsymbol{\varepsilon},r)=\tau_{\boldsymbol{\varepsilon}}(\varepsilon)-r=\sqrt{\bar{\boldsymbol{\sigma}}^+:\mathbf{C}^{e-1}:\bar{\boldsymbol{\sigma}}}-r;$	$\varphi(\boldsymbol{\sigma},q)=\varphi(\bar{\boldsymbol{\sigma}})-(\boldsymbol{\sigma}_y-q)$
Loading-unloading conditions	$g\le 0;\quad \dot{\lambda}\ge 0;\quad \dot{\lambda}g=0;$	$\varphi\le 0;\quad \dot{\lambda}\ge 0;\quad \dot{\lambda}\varphi=0;$
Stress-like internal variable evolution	$\dot{q}=H(r)/\dot{r},\ q\ge 0$ $q\big\vert_{t=0}=r_0=\sigma_u/\sqrt{G};\ q\big\vert_{t=\infty}=r_0$	$\dot{q}=-H/\dot{r},\ q\ge 0$ $q\big\vert_{t=0}=0;\ q\big\vert_{t=\infty}=\boldsymbol{\sigma}_y$
Contitutive tangent Operator	$\boldsymbol{\sigma}=\mathbf{C}^{\tan}:\dot{\boldsymbol{\varepsilon}},$ $\mathbf{C}^{\tan}=\begin{cases}\mathbf{C}^u\equiv(1-d)\mathbf{C}^e=\frac{q}{r}\mathbf{C}^e\\ \mathbf{C}^l\equiv\frac{q}{r}\mathbf{C}^e-\frac{q-Hr}{r^3}\bar{\boldsymbol{\sigma}}^+\otimes\bar{\boldsymbol{\sigma}}\end{cases}$	$\boldsymbol{\sigma}=\mathbf{C}^{\tan}:\dot{\boldsymbol{\varepsilon}},$ $\mathbf{C}^{\tan}=\begin{cases}\mathbf{C}^u\equiv\mathbf{C}^e\\ \mathbf{C}^u\equiv\mathbf{C}^e-\frac{\mathbf{C}^e:\xi\otimes\mathbf{C}^e:\xi}{\xi:\mathbf{C}^e:\xi+\mathbf{H}}\end{cases}$ (5)

where $\Psi(\boldsymbol{\varepsilon},r)$ is the free energy, function of the strains, a variable *r* represents the threshold of initial inelastic behaviour, $\mathbf{C}^e$ is the elastic tensor, d is the damage variable defined in terms of a hardening/softening variable q, dependent on the hardening/softening parameter, H. The variable λ is a damage multiplier, the functions $g(\boldsymbol{\varepsilon},r)$ and $\varphi(\boldsymbol{\sigma},q)$ are the damage surfaces which bound the elastic domain defining the in the strain and stress spaces, respectively; σ_u is the shear strength, $\boldsymbol{\sigma}_y$ is the shear yield stress and G is the shear modulus.

The correctness and the effectiveness of the SD formulation are shown with the results of some representative numerical examples, involving shear failure modes. Based on these results, it is shown that the mesh dependency of the displacement continuum damage models is reduced with the mixed finite element approximation. The overall advantages of a mixed finite element approximation over a displacement formulation are pointed out.

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