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DISCONTINUOUS GALERKIN METHODS AND LOCAL TIME-STEPPING FOR TRANSIENT WAVE MOTION

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ABSTRACT

The accurate and reliable simulation of wave phenomena is of fundamental importance in a wide range of engineering applications such as fiber optics, wireless communication, sonar and radar technology, non-invasive testing and imaging. To address the wide range of difficulties involved, we consider symmetric interior penalty (IP) discontinuous Galerkin (DG) methods, which easily handle elements of various types and shapes, irregular non-matching grids, and even locally varying polynomial order. In [2,3,4] we prove that these methods yield optimal convergence in the L^2-norm and in the enery norm, both for the wave equation and for Maxwell's equations in second-order form. Moreover, in contrast to standard conforming finite element methods, IP-DG methods yield an essentially diagonal mass matrix; hence, when coupled with explicit time integration, the overall numerical scheme remains truly explicit in time.

In the presence of complex geometry, adaptivity and mesh refinement are certainly key for the efficient numerical solution of partial differential equations. However, locally refined meshes impose severe stability constraints on explicit time-stepping schemes, where the maximal time-step allowed by a CFL condition is dictated by the smallest elements in the mesh. When mesh refinement is restricted to a small region, the use of implicit methods, or a very small time step in the entire computational domain, are very high a price to pay. To overcome that stability restriction, we propose local time-stepping schemes, which allow arbitrarily small time steps where small elements in the mesh are located [1]. When combined with a symmetric finite element discretization in space with an essentially diagonal mass matrix, the resulting fully discrete scheme is explicit and exactly conserves a discrete energy. Starting from the standard second order "leap-frog" scheme, we derive time integrators of arbitrary order of convergence. These time-stepping schemes are inspired from symplectic and time reversible integrators for the numerical time integration of Hamiltonian dynamical systems [5].

To illustrate the versatility of our approach, we consider a computational domain that consists of two rectangles connected by a very narrow channel. We use the symmetric IP-DG formulation [4] with P^3 elements on a triangular mesh, which is highly refined inside the narrow region, as shown in the right frame of Fig. 1. Since the typical mesh size inside the refined region is about $p = 17$ times smaller than in the surrounding coarse region, we take p local time steps of size $\Delta\tau = \Delta t/p$ for every time step Δt.

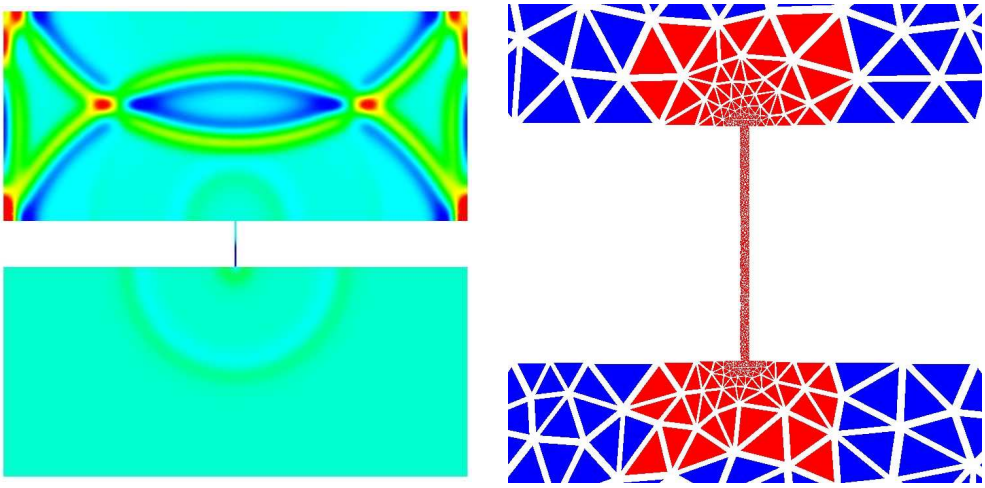

Figure 1: Left: The solution at time $t = 0.2$. Right: the highly refined mesh inside the narrow channel.

When the "fine" region, where local time-steps are used, slightly extends into the surrounding "coarse" region of the mesh, we find that the resulting numerical scheme permits the use of the optimal maximal time-step, dictated by the coarse mesh size.

As shown in Fig. 1, the wave is initiated by a pulse in the upper region, which propagates outward until it impinges on the boundaries. A fraction of the wave then penetrates the channel and generates a circular outgoing wave as it reaches the opposite lower region. Further reflections occur as the wave moves back and forth inside the channel, subsequently generating multiple circular waves in the upper and lower domains.

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