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ON THE APPROXIMATION OF THE SUBGRID SCALE FOR SYSTEMS OF EQUATIONS

*** Javier Principe[1] and Ramon Codina[1]**

[1] Universitat Politècnica de Catalunya
Jordi Girona 1-3, Edifici C1, 08034 Barcelona
principe@cimne.upc.edu and ramon.codina@upc.edu

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ABSTRACT

We consider a second order system of n equations for n unknowns $\boldsymbol{U}$ of the form

$$\mathcal{L}\boldsymbol{U} := -\partial_i\left(\boldsymbol{K}_{ij}\partial_j\boldsymbol{U}\right) + \partial_i\left(\boldsymbol{A}_i\boldsymbol{U}\right) + \boldsymbol{S}\boldsymbol{U} = \boldsymbol{F} \quad \text{in} \quad \Omega \tag{1}$$

where Ω is an open domain in $\mathbb{R}^{\mathrm{d}}$, $\boldsymbol{K}_{ij}, \boldsymbol{A}_i$ and $\boldsymbol{S}$ (for$1 \leq i,j \leq \mathrm{d}$) are square coefficient matrices of $n \times n$ components and $\mathrm{d} = 2,3$ is the number of space dimensions, supplemented with appropriate boundary and initial conditions. Several systems can be written in this form and we will pay particular attention to the important case of the incompressible Navier Stokes equations. The Galerkin approximation of this problem may be unstable in several ways and this depends on the system under consideration. We consider the application of the variational multiscale formulation of [1] that is based on the decomposition of the space where the problem is posed $\boldsymbol{\mathcal{V}}$ as $\boldsymbol{\mathcal{V}}_h \oplus \boldsymbol{\mathcal{V}}'$ where $\boldsymbol{\mathcal{V}}_h$ is the finite element space and $\boldsymbol{\mathcal{V}}'$ the subgrid scale space. This decomposition leads to a modified finite element problem for the finite element component $\boldsymbol{U}_h$ that depends on the subgrid scale $\boldsymbol{U}'$ in a way that provides stability and to a fine scale problem. The fine scale problem is defined on each element K as (the uncoupling of elemental problems is the first approximation performed)

$$\mathcal{L}\boldsymbol{U}' = \boldsymbol{F} - \mathcal{L}\boldsymbol{U}_h := \boldsymbol{R} \tag{2}$$

and the objective of this work is to discuss some general ideas for its approximated solution.

Our argument is an extension of the heuristic Fourier analysis performed in [2] for the scalar case and a main point is to impose the approximated solution $\boldsymbol{U}'_{\mathrm{ap}}$ of 2 to bound the norm of the residual in the same way the exact solution $\boldsymbol{U}'$ does. To do that we need to define an appropriate norm on the space $\boldsymbol{\mathcal{V}}$ and to compare forces we also need to define a norm in the space $\boldsymbol{\mathcal{W}} := \mathcal{L}\left(\boldsymbol{\mathcal{V}}\right)$. In general neither $\boldsymbol{U}^t\boldsymbol{U}$ nor $\boldsymbol{F}^t\boldsymbol{F}$ are dimensionally meaningful (only the product $\boldsymbol{U}^t\boldsymbol{F}$, that represents the work done by $\boldsymbol{U}$ against $\boldsymbol{F}$, is assumed to be dimensionally consistent). Therefore we introduce a symmetric and positive-definite scaling matrix $\boldsymbol{M}$ such that the product $\left(\boldsymbol{F}_1, \boldsymbol{F}_2\right)_M := \boldsymbol{F}_1^t\boldsymbol{M}\boldsymbol{F}_2$ is pointwise well defined and we define the corresponding norm as $|\cdot|_M$ and by $\|\cdot\|_M$ we denote the $L^2\left(\Omega\right)$-norm of $|\cdot|_M$. The choice of this scaling is equivalent to choose the way the equations are written in dimensionless form, if this is the option adopted.

The first step we perform is to transform equation 2 to a reference domain $\widehat{K}$, what allows us to determine the dependence with respect to the grid size h. If we consider the simple case of undeformed elements this operation results in a redefinition of $\mathcal{L}$ changing $\boldsymbol{K}_{ij}$ by $h^{-2}\boldsymbol{K}_{ij}$ and $\boldsymbol{A}_i$ by $h^{-1}\boldsymbol{A}_i$. The second step is to Fourier transform 2 (on the reference domain) to obtain

$$\boldsymbol{S}(k)\,\widehat{\boldsymbol{U}'} := \left(h^{-2}k_ik_j\boldsymbol{K}_{ij} + \mathrm{i}h^{-1}k_i\boldsymbol{A}_i + \boldsymbol{S}\right)\widehat{\boldsymbol{U}'} = \widehat{\boldsymbol{R}}$$

where $\mathrm{i} = \sqrt{-1}$. Taking the M-norm of this complex valuated *algebraic* equation it is possible to show that if we assume $\boldsymbol{U}'_{\mathrm{ap}} = \boldsymbol{\tau}\boldsymbol{R}$ where $\boldsymbol{\tau}$ is a symmetric and positive definite matrix, the same bound of the residual is obtained provided $\left|\boldsymbol{\tau}^{-1}\right|_M = \left|\boldsymbol{S}\left(k^0\right)\right|_M$ for some k^0 whose existence is guaranteed by the mean value theorem. Note that this condition is satisfied if the largest eigenvalues of $\boldsymbol{\tau}^{-1}\boldsymbol{M}\boldsymbol{\tau}^{-1}$ and $\boldsymbol{S}\left(k^0\right)\boldsymbol{M}\boldsymbol{S}\left(k^0\right)$ coincide, which is in particular obtained when the spectrum with respect to $\boldsymbol{M}^{-1}$ of $\boldsymbol{\tau}^{-1}\boldsymbol{M}\boldsymbol{\tau}^{-1}$ and of $\boldsymbol{S}\left(k^0\right)\boldsymbol{M}\boldsymbol{S}\left(k^0\right)$ are the same. As we design the stabilization matrix paying attention to the simplicity of the final formulation (in general we take $\boldsymbol{\tau}$ diagonal) this optimal situation is not always possible.

We have applied this strategy to design the matrix of stabilization parameters for different differential systems, such as the $\boldsymbol{\sigma}$-$\boldsymbol{u}$-p formulation of the elasticity problem, the mixed approximation of the wave equation, the Boussinesq shallow water equations and the incompressible Navier Stokes equations. In the particular case of the Oseen equations, given by

$$\begin{aligned} -\nu\nabla^2\boldsymbol{u} + \boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}p &= \boldsymbol{f} \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 \end{aligned}$$

where ν is the dynamic viscosity and $\boldsymbol{a}$ is the advection field, assuming $\boldsymbol{\tau}$ diagonal we obtain (in 2D) $\boldsymbol{\tau} = \mathrm{diag}(\tau_m, \tau_m, \tau_c)$ where $\tau_m = [(c_1\nu/h^2)^2 + (c_2|\boldsymbol{a}|/h)^2]^{1/2}$ and $\tau_c = c_1h^2/\tau_m$.

However, for this problem we can obtain a better approximation of the subscales. Using the previous argument we also obtain τ_m as an approximation to the scalar convection diffusion operator. Then, taking the divergence of the momentum equation we arrive to a Poisson equation for the pressure and we approximate the inverse of the Laplace operator by c_1h^2. Replacing into the momentum equation and approximating the convection diffusion operator by τ_m we obtain a better approximation for the subscales

$$\boldsymbol{u}' = \tau_m\boldsymbol{R}_m + c_1h^2\tau_m\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{R}_m\right) - c_1h^2\boldsymbol{\nabla}R_c \quad (3)$$

$$p' = -c_1h^2\boldsymbol{\nabla}\cdot\boldsymbol{R}_m + c_1h^2\tau_m^{-1}R_c \quad (4)$$

where $\boldsymbol{R}_m$ is the residual of the momentum equation and R_c the residual of the continuity equation. This solution can be also obtained computing the inverse of $\boldsymbol{S}(k)$, approximating the inverse of the scalar operator $\left(-\nu k^2 + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}\right)$ by τ_m and applying the inverse Fourier transform. As the coupling terms in $\boldsymbol{S}(k)^{-1}$ are kept approximating only the convection diffusion and Laplace operators, we finally obtain a stabilization operator instead of a stabilization matrix. The final approximation obtained using the model 3 and 4 is shown to be stable.

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