8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

Numerical solutions for optimal control of the bidomain equations

*** Chamakuri Nagaiah[1], Karl Kunisch[1] and Gernot Plank[2]**

[1] Institute of Mathematics and Scientific Computing, University of Graz
Heinrichstr. 36, Graz, A-8010, Austria.
nagaiah.chamakuri@uni-graz.at,
karl.kunisch@uni-graz.at

[2] Institute of Biophysics, Medical University of Graz
Harrachgasse 21, Graz, A-8010, Austria.
gernot.plank@meduni-graz.at

Key Words: *Electrical activity in cardiac tissue, Bidomain model, Reaction-diffusion systems, Optimal control system, PDE constraints, FEM, Linearly implicit RK methods.*

ABSTRACT

In this work we present the numerical solution of optimality systems corresponding to optimal control problems governed by the bidomain equations which are widely used for describing the electrical activity of the cardiac tissue. This bidomain model is based on partial differential equations of elliptic and parabolic type, and a system of stiff ordinary differential equations. The space discretization of the state and dual variables is achieved by the conforming finite element method, and the time discretization is based on linearly implicit time stepping methods. The goal of this work is to present a first step towards optimal control of the bidomain equations with the transmembrane current density stimulus acting as the control variable and the control objective consisting in suppressing arrhythmia.

The bidomain equations: The most complete description of cardiac electricity is given by the bidomain equations, where Ω_c denotes the cardiac tissue sample domain and we set $Q_c = \Omega_c \times [0, t_f]$.

$$-\nabla \cdot (\bar{\sigma_i} + \bar{\sigma_e})\nabla\phi_e - \nabla \cdot \bar{\sigma_i}\nabla V_m = I_e \quad \text{in } Q_c \tag{1}$$

$$\nabla \cdot \bar{\sigma_i}\nabla V_m + \nabla \cdot \bar{\sigma_i}\nabla\phi_e = \beta\left(C_m\frac{\partial V_m}{\partial t} + I_{ion}(V_m, v) - I_{tr}\right) \quad \text{in } Q_c \tag{2}$$

$$\frac{\partial v}{\partial t} = g(V_m, v) \quad \text{in } Q_c \tag{3}$$

ϕ_i and $\phi_e : Q_c \to \mathbb{R}$ are the intracellular and extracellular potentials, $V_m : Q_c \to \mathbb{R}$ is the transmembrane voltage, $v : Q_c \to \mathbb{R}^n$ represents the ionic current variables, $\bar{\sigma_i} : \Omega_c \to \mathbb{R}^{d\times d}$ and $\bar{\sigma_e} : \Omega_c \to \mathbb{R}^{d\times d}$ are respectively the intracellular and extracellular conductivity tensors, β is the surface to volume ratio of the cardiac cells, I_{tr} is the transmembrane current density stimulus as delivered by the intracellular electrode, I_e is an extracellular current density stimulus, C_m is the capacitance per unit area, and I_{ion} is the current density flowing through the ionic channels. Eqn. (1) is an elliptic equation and Eqn. (2) is a parabolic equation. Eqn. (3) is a set of non linear ordinary differential equations which can be solved independently for each node. Assume that the intracellular and extra cellular potentials have homogeneous Neumann boundary conditions. The initial values of the transmembrane voltage

and state variables are prescribed by given constant values. Here $\partial Q_c = \partial\Omega_c \times [0, t_f]$. Furthermore, for the model equations, conductivity coefficients and discussion on treatment of boundary conditions, we refer to R. W. Santos et. al [1]. The ionic model currents and the cell membrane model, we considered, are based on the following Fitzhugh-Nagumo variant, for details see Colli Franzone et. al [2],

$$I_{ion}(V_m, v) = GV_m(1 - \frac{V_m}{v_{th}})(1 - \frac{V_m}{v_p}) + \eta_1 V_m v\,, \quad g(V_m, v) = \eta_2(\frac{V_m}{v_p} - \eta_3 v) \tag{4}$$

where $G, \eta_1, \eta_2, \eta_3$ are positive real coefficients, v_{th} is a threshold potential and v_p the peak potential.

Optimality system: The choice of the cost functional which is suitable to optimize the potentials and currents given by

$$J(V_m, I_{tr}, I_e) = \min \frac{1}{2} \int_0^T \left(\int_{\Omega_c} |\nabla V_m|^2 \, \mathrm{d}\Omega_c + \alpha \int_{\Omega_c} (|I_{tr}|^2 + |I_e|^2) \mathrm{d}\Omega_c \right) dt \tag{5}$$

After construction of the Lagrangian, the optimality equations can be derived setting the first order variation with respect to ϕ_e, V_m and v equal to zero. This leads to

$$-\nabla \cdot (\bar{\sigma_i} + \bar{\sigma_e})\nabla p + \nabla \cdot \bar{\sigma_i} \nabla q = 0\,, \tag{6}$$

$$-\nabla \cdot \nabla V_m - \nabla \cdot \bar{\sigma_i} \nabla p + \nabla \cdot \bar{\sigma_i} \nabla q + \beta(C_m q_t - (I_{ion})_{V_m} q) - g_{V_m} \zeta = 0\,, \tag{7}$$

$$-\beta (I_{ion})_v q - \zeta_t - g_v^T(V_m, v)\zeta = 0\,. \tag{8}$$

The Optimality conditions are $\alpha I_{tr} + \beta q = 0\,,\ \alpha I_e - p = 0$, the terminal conditions are $\zeta(T) = 0\,,\ q(T) = 0$ and the boundary conditions are $\sigma_e \bar{+} \sigma_e \nabla p \cdot \eta = 0 \quad \text{on } \partial Q_c\,,\ \bar{\sigma_i} \nabla q \cdot \eta = 0 \quad \text{on } \partial Q_c$.

Numerical discretization and results: The spatial discretization of the optimality system is based on piecewise linear finite elements, while the temporal discretization uses linearly implicit Runge-Kutta methods. First results are available for conjugate gradient based optimization strategies. Second order methods are currently being developed.

Conclusions: The presented numerical methods motivate us to study further the behavior of reaction-diffusion systems coupled with non linear stiff ordinary differential equations as they arise in cardiac modeling. The gradient in the cost leading to rough solutions of the adjoint equation and the pattern formation type behavior of the dynamical system imply many challenging challenges to efficient numerical implementations. To cope with long time horizons, receding horizon strategies are currently being tested as well.

Acknowledgement: The authors gratefully acknowledge the Austrian Science Fund FWF for financial support and this work was supported in the framework of the SFB programm "Mathematical Optimization and Applications in Biomedical Sciences".

REFERENCES

[1] R. Weber dos Santos, G. Plank, S. Bauer, EJ. Vigmond, "Parallel multigrid preconditioner for the cardiac bidomain model", *IEEE Trans Biomed Eng.*, Vol. **51(11)**, 1960-1968, 2004.

[2] P. Colli Franzone, P. Deuflhard, B. Erdmann, J. Lang, L. Pavarino, "Adaptivity in Space and Time for Reaction-Diffusion Systems in Electrocardiology", *SIAM J. Sc. Comp.*, Vol. **28**, 942-962, 2006.