8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

Stabilized Discontinous Galerkin Formulations for Ocean Modelling

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Key Words: *Ocean Modelling, Finite Elements, Discontinuous Galerkin Methods, Stabilized Dormulations, Partial Differential Equations on Sphere, Dispersion Analysis*

ABSTRACT

Finite Element methods are highly compelling for numerical ocean modeling. On one hand, complex coastline shapes can faithfully be represented by locally increasing the mesh resolution without any constraint on the mesh topology. On the other hand, the systematic use of local coordinates allows to avoid the classical singularity problem occurring at both poles with structured meshes.

The inviscid shallow water equations can be obtained by averaging the incompressible Navier-Stokes equations in the vertical direction. The usual non-conservative form reads:

$$\begin{aligned} \mathbf{u}_{,t} + \mathbf{u}\cdot\nabla\mathbf{u} + f\mathbf{k}\times\mathbf{u} + g\nabla\eta &= 0 \\ \eta_{,t} + \nabla\cdot[(h+\eta)\mathbf{u}] &= 0, \end{aligned}$$

where $\mathbf{u}$ is the two-dimensional mean velocity, η is the elevation of the free-surface, f is the Coriolis parameter, $\mathbf{k}$ is a unit upward normal vector, g is the gravitational acceleration and h is the local reference depth at rest. Extension to three-dimensional hydrostatic calculations will also be presented in the final applications.

Our ocean model uses several efficient mixed finite element pairs for the primitive shallow-water equations that did not support spurious oscillations. In particular, we developed an original mixed formulation based the $P^1_{NC} - P^1$ finite element pair [1][2]. This pair is a good compromise between continuous and discontinuous Galerkin methods, and appears to behave rather well for shallow water flows. Moreover, the model consistently conserves mass and tracers [3].

In this talk, we firstly address the issue to solve problems on the sphere (and even on any curved geometries, in a more general sense). Any global coordinates system cannot be used, since it introduces poles and will generate singularity for the representation of all fields at poles. Then, we compare, in terms of efficiency and accuracy, the following element pairs $P^1 - P^1$, $P^1_{NC} - P^1$, $P^1_{disc} - P^1$ and $P^1_{disc} - P^1_{disc}$. Typically, some pairs look very attractive as they provide the right compromise between continous and discontinous Galerkin methods. However, it does not allow to introduce a straighforward application of

the Riemann solvers used in full Discontinuous Galerkin methods. Therefore, it is mandatory to compare what element is the more robust, more efficient and more accurate. Finally, the impact of adding some subgrid viscosity is also highlighted and appears to be a quite critical factor in this analysis.

We present some validation results with the benchmark test cases described by Williamson et al. [4]. It consists on idealized, but quite realistic non-viscous flows on the sphere. We are able to circumvent the singularity problems inherent to global coordinate systems typically encountered. We demonstrate that more accurate and more stable results can be obtained with the stabilized version of all mixed formulations. The dispersion properties are the last major criteria. In order to perform such an analysis, we derive a reference numerical dispersion relation with a variable Coriolis parameter. Then we discretize the equations with the Discontinuous Galerkin method and find discrete dispersion relations by developing a general grid-independent modal analysis. We successfully compare this discrete dispersion relation with the reference solution and analyze the dispersion and dissipation errors [5]. Finally, typical applications of the SLIM model applied to the global ocean or some regional flows will demonstrate the current state of the art in terms of oceanic modelisation with unstructured grids.

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