Conservative upwind finite-element method for the Keller-Segel system of chemotaxis

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In 1970, F. F. Keller and L. A. Segel (cf. [4]) proposed the system of partial differential equations that described the aggregation of slime molds resulting from their chemotactic features. The system is now called the *Keller-Segel system modelling chemotaxis*, and a large number of works are devoted to mathematical analysis to the system (cf. [2], [3], and [7]). In this paper, we consider a variant of the Keller-Segel system:

$$\begin{cases} u_t = \nabla\cdot(D_u\nabla u - u\nabla\phi(v)) & \text{in} \quad \Omega\times(0,T), \\ kv_t = D_v\Delta v - k_1 v + k_2 u & \text{in} \quad \Omega\times(0,T), \\ \dfrac{\partial u}{\partial\nu}=0, \quad \dfrac{\partial v}{\partial\nu}=0 & \text{on} \quad \partial\Omega\times(0,T), \\ u|_{t=0}=u_0(x), \quad v|_{t=0}=v_0(x) & \text{on} \quad \Omega, \end{cases} \tag{KS}$$

where $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded domain with the boundary $\partial\Omega$; $\partial/\partial\nu$ represents differentiation along ν on $\partial\Omega$; ν is the outer unit normal vector to $\partial\Omega$; $u=u(x,t)$ is the density of the cellular slime molds and $v=v(x,t)$ the concentration of the chemical substance; the chemotactical sensitivity function $\phi(v)$ is assumed to be nondecreasing and smooth on $v>0$ (e.g. $\phi(v)=\lambda v$, $\phi(v)=\lambda\log v$ and $\phi(v)=\lambda v^2/(\lambda'+v^2)$ with $\lambda,\lambda'>0$); D_u and D_v are diffusion coefficients, k is the relaxation time, k_1 and k_2 correspond the generation rate (they are assumed to be positive constants); the initial values $u_0(x)$ and $v_0(x)$ are assumed to be smooth, ≥ 0 and $\not\equiv 0$.

We have dual purpose. The first is to propose a finite-element scheme that satisfies the conservations of positivity and total mass of the solution. Thus, our finite-element solution preserves the value of the L^1 norm, which is the discrete version of an important property of the original system. The scheme makes use of Baba-Tabata type upwind finite-element approximation (cf. [1]) and semi-implicit time discretization with step-size control (cf. [6]). That is, at every discrete time step $t_n=\tau_1+\cdots+\tau_n$, we adjust the time increment τ_n to obtain a positive solution. Consequently, we realize the L^1 conservative finite-element approximation for an arbitrary $h > 0$, the granularity parameter of the spatial discretization. Moreover, our scheme is well suited for practical computations.

The second purpose is to develop an error analysis. Because of the nonlinear feature of the system and the application of Baba-Tabata's upwind approximation, it is crucial to study the error in the $L^p \times W^{1,\infty}$ norm (uniformly in t_n) for some $p > d$. Therefore, we shall develop the L^p theory for (KS) and its finite element approximation. Thus, we formulate (KS) and its finite element approximation as abstract evolution equations of parabolic type in Banach spaces. Then, we apply analytical (holomorphic) semigroup theory. Actually, if the triangulation is of acute type, the operator A_h, a finite element approximation of $-\Delta + 1$ of the lumped mass type, becomes sectorial on a finite element space $\mathscr{X}_{h,p}$ equipped with a modified L^p norm. In particular, $-A_h$ generates an analytic semigroup on $\mathscr{X}_{h,p}$. (The precise meaning of these symbols will be given in the talk; See [5].) We then make use of Duhamel's principle, fractional powers of operators, and the smoothing property of the semigroup. Although semigroup theory is somewhat abstract, several L^p estimates can be derived in a quite formal manner. Furthermore, our method of analysis is a discrete analogue of the standard approach for solving nonlinear evolution problems. It also provides a theoretical framework for the approximation error analysis of some other nonlinear evolution problems.

References

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