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THE BOUNDARY ELEMENT COLLOCATION METHOD FOR A FRACTIONAL DIFFUSION EQUATION

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ABSTRACT

The fractional diffusion equation is a generalization of diffusion equation via fractional time derivatives. The time-fractional diffusion equation underlies Levy random walks and fractional brownian motion which finds their applications in mathematical physics and finance, where one analysis the stock price changes and interest rate changes [1], [3].

The fractional diffusion equation on $\Omega \times (0,T)$, where Ω is a bounded smooth domain in $\mathbb{R}^2$ and $T > 0$, reads as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - D\Delta u(x,t) = 0,\ 0 < \alpha \le 1,\ (x,t) \in \Omega \times (0,T), \tag{1}$$

where Δ is the Laplace operator, D denotes a physical constant and $\frac{\partial^\alpha u}{\partial t^\alpha}$ denotes the fractional order time derivative of order α (in the Caputo sense). We recognize that the fractional diffusion equation is equivalent to the integro-partial differential equation

$$u(x,t) - \frac{D}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \Delta u(x,t) d\tau = 0. \tag{2}$$

Such integro-partial differential equations have been recently investigated by several authors [4], [5].

We assume throughout the paper that the solution has the following inital-boundary conditions:

$$\begin{aligned} u|_{\Sigma_T} &= g, \\ u(x,0) = 0, \end{aligned}$$

where $\Sigma_T = \partial\Omega \times (0,T)$ is the cylindrical boundary of the domain.

The fractional diffusion equation can be reformulated as a boundary integral equation using the fundamental solution of the fractional diffusion equation that is the Fox H-function

$$E(x,t) = \frac{1}{\pi} t^{\alpha-1} |x|^{-2} H_{12}^{20}(|x|^2 t^{-\alpha}|_{(1,1),(1,1)}^{\alpha,\alpha}.$$

We apply the indirect approach where the solution is represented as a single layer potential. Using the continuity of the monopole potential up to the boundary we derive the equivalent boundary integral equation

$$V\psi(x,t) = \int_{\partial\Omega} \int_0^t \psi(y,t) E(x-y, t-\tau) ds_y d\tau = g(x,t). \tag{3}$$

We discretize the boundary integral equation using the spline collocation method. For the approximation we use tensor product splines such that the spatial variable is approximated by 1-periodic smoothest splines $S_{h_x}^{d_x}$ of degree $d_x \in \mathbb{N}_0$, but for the time discretization we use only the piecewise linear or constant splines $\tilde{S}_{h_t}^{d_t}$ for which the support $\mathrm{supp}(v) \subset [0,\infty)$ as in [2].

Finally, we shall provide the stability and convergence results in the approriate Sobolev spaces. We will prove the following optimal order error estimates: *The collocation solution of (3) satisfies the asymptotic error estimates:*

$$\|u - u_h\|_{H^t_{\frac{2}{\alpha}}} \le C h_x^{s-t} \|u\|_{H^s_{\frac{2}{\alpha}}},$$

where $-1 \le t < \min\{d_x, 2d_t + \frac{1}{2}\} + \frac{1}{2}$, $\frac{1}{2} < s < \min_{\{} d_x, 2d_t + 1\}$. *and*

$$0 < h_x^2 h_t^{-\alpha} < \infty.$$

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