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NONLINEAR MEMBRANE LOCKING-FREE CURVED-BEAM ELEMENT FOR ARCHES

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ABSTRACT

This paper presents a new membrane-locking free curved-beam element for the nonlinear in-plane large deformation analysis of arches. In the conventional formulation of curved-beam elements, the strains under in-plane loading consist of the membrane strain and the bending strain and given by

$$\epsilon_{ss} = \epsilon_m + \epsilon_b \ \text{ with } \ \epsilon_m = w' - v/R \ \text{ and } \ \epsilon_b = -y\kappa = -y\left(v'' + w'/R\right), \tag{1}$$

where v and w are displacements in the radial axis oy and axial axis os (Figure 1), κ is the curvature change due to deformation, and $(\ )' \equiv \mathrm{d}(\ )/\mathrm{d}s$.

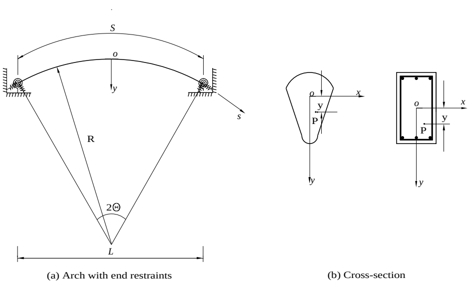

Figure 1: Restrained arch.

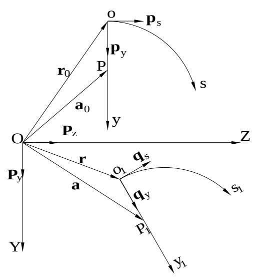

Figure 2: Axes and position vectors.

The axial and radial displacements can be expressed as

$$w = a_1 + a_2 s, \quad v = a_3 + a_4 s + a_5 s^2 + a_6 s^3 \tag{2}$$

where a_i are generalized degree of freedoms. Substituting Eqn (2) into Eqn (1) leads to

$$\epsilon_m = a_2 - \left(\frac{a_3}{R} + \frac{a_4 s}{R} + \frac{a_5 s^2}{R} + \frac{a_6 s^3}{R}\right) \quad \text{and} \quad \kappa = 2a_5 + 6a_6 s - \frac{a_2}{R}. \tag{3}$$

Under most conditions of loading, a slender arch bends but has very little membrane strain. In the limit of slenderness, the membrane strain vanishes, which is known as the inextensibility condition. The inextensibility condition $\epsilon_m = 0$ for all s requires that

$$a_2 - a_3/R = 0, \;\; a_4 = a_5 = a_6 = 0. \tag{4}$$

If the condition $a_2 - a_3/R = a_5 = a_6 = 0$ is enforced, then the only contribution to the curvature κ comes from the membrane term a_2. The nonzero a_i produces nonzero ϵ_m whose associated strain energy and stiffness are very large for a slender element. Thus, when a bending load is applied, bending deformation tends to be "locked out" of the element response. In this case, the membrane and bending strains given by Eqn (1) interact unfavourably in the curved element, so that nodal displacements that should be resisted only by bending are resisted by membrane deformation as well. Because membrane stiffness is far greater than bending stiffness is a slender arch, the desired bending mode tends to be excluded from element response to load.

In this paper, an exact rotation formulation is used to derive the strains, The rotations from the basis vectors $\mathbf{p}_y, \mathbf{p}_s$ in the undeformed configuration to the basis vectors $\mathbf{q}_y, \mathbf{q}_s$ in the deformed configuration can be described as

$$\{\mathbf{p}_y, \mathbf{p}_s\}^T = \mathbf{R}\{\mathbf{q}_y, \mathbf{q}_s\}^T \;\; \text{with} \;\; \mathbf{R} = \begin{bmatrix} (1+\tilde{w}')/(1+\epsilon) & \tilde{v}'/(1+\epsilon) \\ -\tilde{v}'/(1+\epsilon) & (1+\tilde{w}')/(1+\epsilon) \end{bmatrix}. \tag{5}$$

where $(1+\epsilon) = \sqrt{(1+\tilde{w}')^2 + \tilde{v}'^2}$, $\tilde{v}' = v' - w\kappa_0$, $\tilde{w}' = w' + v\kappa_0$, and $\kappa_0 = -1/R$. The matrix $\mathbf{R}$ is a skew-symmetric and satisfies the orthogonal and unimodular conditions of the two dimensional special orthogonal group SO(2) that $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$ and det$\mathbf{R} = 1$.

From the position vector analysis, the strain ϵ_{ss} can be obtained as

$$\epsilon_{ss} = \tilde{w}' + \frac{1}{2}\tilde{v}'^2 + \frac{1}{2}\tilde{w}'^2 - y\left\{\tilde{v}''(1+\tilde{w}') - \tilde{v}'\tilde{w}'' + \kappa_0[(1+\tilde{w}')^2 + \tilde{v}'^2]^{1/2} - \kappa_0\right\}, \tag{6}$$

where all the higher order terms are retained, and the term $\kappa_0[(1+\tilde{w}')^2 + \tilde{v}'^2]^{1/2}$ includes the interaction of the axial extension with the initial curvature on the curvature changes. Substituting $\tilde{v}' = v' - w\kappa_0$, $\tilde{w}' = w' + v\kappa_0$ into Eqn (6) and ignoring the higher order terms leads to

$$\epsilon_{ss} \approx w' + v\kappa_0 + \frac{1}{2}(v' - w\kappa_0)^2 - y(v'' + v\kappa_0^2) \tag{7}$$

which shows that the curvature change is given by $\kappa = v'' + v\kappa_0^2$ and so the axial deformations in the strain given by Eqn (6) do not affect the curvature change. Hence, the curved beam element based on the nonlinear strain given by (6) will be membrane locking-free.

By comparing the curvature change given by Eqn (7) with that given by Eqn (1) and noting $\kappa_0 = -1/R$, it can be obtained that

$$v'' + v\kappa_0^2 = v'' - w'\kappa_0 \;\Rightarrow\; w' + v\kappa_0 = 0 \;\Rightarrow\; \epsilon_m = w' - \frac{v}{R} = 0 \tag{8}$$

which indicates the required axial inextension condition is satisfied.

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