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ON THE CONVERGENCE OF PATH-CONSERVATIVE SCHEMES FOR NONCONSERVATIVE PROBLEMS

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ABSTRACT

This work is concerned with the numerical approximation of Cauchy problems for one-dimensional hyperbolic nonconservative systems. Some particular cases of P.D.E. systems which can be considered as particular cases of the nonconservative systems that we study are systems of conservation laws, systems of conservation laws with source term or balance laws and coupled systems of conservation laws as defined in [1]. Such systems occur in many applications. For instance, the shallow water systems that govern the flow of one shallow layer or two superposed shallow layers of immiscible homogeneous fluids can be written, respectively, as a balance law or a coupled system of two balance laws.

The main difficulty of nonconservative systems from both the mathematical and the numerical point of view comes from the presence of nonconservative products, which do not make sense as distributions when the solution is discontinuous. In this work we assume the definition of nonconservative products as Borel measures given by Dal Maso, LeFloch and Murat [2]. This definition depends on the choice of a family of paths in the phase space and allows to give a rigorous definition of weak solutions of the noconservative system, which generalizes the usual concept of weak solutions of systems of conservation laws. Together with the definition of weak solutions, a notion of entropy has to be chosen, as the usual Lax's concept or one related to an entropy pair. Once this selection has been done, the classical theory of simple waves of hyperbolic systems of conservation laws and the results concerning the solutions of Riemann problems can be extended to noconservative systems.

The choice of an appropriate family of paths may be a difficult task. Although in physical applications it should depend on the problem under consideration and it could be suggested by an argument of regularization, it is natural from the mathematical point of view to impose this family to satisfy some conditions concerning the relationship of the paths with the integral curves of characteristic fields (see [5]).

Once the family of paths has been chosen, the problem is how to choose a numerical scheme in such a way that, if the approximations produced by the scheme converge to some function as the mesh is

refined, then this function is a weak solution of the system in the sense given by that family of paths. In the particular case of systems of conservation laws, the classical theorem of Lax-Wendroff [4] gives the answer to this problem: conservative numerical methods have this property. Moreover, in [3] some negative results have been shown concerning the failure of the convergence of nonconservative schemes to weak solutions.

The main goal of this work is to prove that the family of path-conservative numerical schemes introduced in [6] has a similar property for the noconservative case. This concept, which is a generalization of that of conservative scheme for systems of conservation laws, is also related to the choice of a family of paths. In that article, it was also introduced a notion of path-conservative Approximate Riemann Solver to produce path-conservative numerical schemes. Based on this notion, Godunov, Roe or Relaxation methods can be constructed for the nonconservative systems that we study.

In [8] a Lax-Wendroff type theorem has been proved for a class of well-balanced numerical schemes for solving scalar conservation laws with a source term, while in [5] a convergence theorem has been proved for a Godunov method for nonconservative systems and it has been extended to a family of numerical schemes based on Approximate Riemann Solvers. Nevertheless, as far as we know, there is not a previous result of this nature for more general schemes for noconservative systems. We prove the convergence result conjectured in [6]: if the approximations provided by a path-conservative scheme converge in a certain manner (uniformly in the sense of graphs [2]) to some function as the mesh is refined, then this function is a weak solution of the nonconservative system if both the definitions of path-conservative scheme and weak solution make reference to the same family of paths. Nevertheless, this result cannot be considered as an extension of the classical Lax-Wendroff theorem for conservative problems, as the convergence required here is stronger than that required in the classical result, and it must be understood as a consistency result of the chosen numerical schemes with the chosen notion of weak solutions. Further investigation based on the use of weaker hypotheses is being developed by the authors.

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