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MATHEMATICAL MODELLING OF SELF CONTACT IN HYPERELASTIC RODS

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ABSTRACT

In the last three decades, the theory of elastic rods has witnessed great development because of its various industrial and biomechanical applications. Among the types of elastic rods used in practical applications, we cite, beams in civil constructions, cables in marine industries, pipelines in the oil industries, and fragments of the DNA molecule in the modeling of live sciences. The field of open problems in this context is still very vast. Motivated by the application of elastic rods in the modeling of deformations of the DNA molecule, we propose to study the mathematical and numerical aspects of the equilibrium configurations of an elastic rod, subject to terminal loads and which might contains points or regions of self-contact not known a priori. We describe a uni-dimensional model for the treatment of self-contact in elastic rod that is based on the Cosserat theory. The particularity of the model we propose is the introduction of a simplified contact distance that takes into account the geometric description of the rod. We first note that the computation of the contact distance is equivalent to an orthogonal projection computation of a point on a curve. By minimizing the elastic energy under the constraint that the contact distance is non-positive we avoid self-penetration of different parts of the rod. We have made some numerical simulations using a four-node curved finite element and a multiplicative method for updating rotations as described by Simo & Vu-Quoc [3].

We consider an elastic rod $\mathcal{R}$ of length L and circular cross sections of a uniform diameter 2ε. The configuration of the rod is described by specifying, for each $s \in [0, L]$, a position vector $\boldsymbol{r}(s)$ and a right-handed triad of orthonormal directors $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$. The curve $\mathcal{C} \equiv \{\boldsymbol{r}(s), s \in [0, L]\}$ represents the line of centroids of the cross-sections, in the deformed configuration of $\mathcal{R}$. The triad $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$ gives the orientation of the material cross section at s of $\mathcal{R}$. We take $\boldsymbol{R}(s)$ to denote the proper orthogonal tensor $\boldsymbol{R}(s) = \boldsymbol{d}_i(s) \otimes \boldsymbol{e}_i$.

The kinematics of the rod are encapsulated in the following two equations

$$\boldsymbol{r}'(s) = \boldsymbol{v}(s), \boldsymbol{R}'(s) = \boldsymbol{u}^{\times}(s)\boldsymbol{R}(s),$$

Let $\boldsymbol{n}(s)$ (respectively $\boldsymbol{m}(s)$) be the resultant force (respectively couple) acting across the section at s of $\mathcal{R}$ about the point $\boldsymbol{r}(s)$. The local form of the balance of forces and moments for the rod write [1]

$$\boldsymbol{n}' + \boldsymbol{f} = \boldsymbol{0}, \tag{1}$$

$$\boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} + \boldsymbol{t} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{f}$ and $\boldsymbol{t}$ are external distributed force and torque per unit length of the reference configuration.

As self-penetration of different parts of the rod is physically impossible. To prevent it, we shall impose the constraint $d_{\boldsymbol{r}}(s) \leq 0,\ \forall s \in [0, L]$, where we define at each s a distance of contact by the distance between the points on the central curve of the rod by

$$d_{\boldsymbol{r}}(s) = 2\varepsilon - \min_{\sigma \in I(s)} \|\boldsymbol{r}(s) - \boldsymbol{r}(\sigma)\|. \tag{3}$$

$I(s)$ is a subset of $[0, L]$ which is selected according to the global curvature $\mathcal{R}[\boldsymbol{r}]$ of the rod [2], associated with the portion $\{\boldsymbol{r}(\tau), \quad \tau \in [0, L] \setminus I(s)\}$ of the curve in which the contact can take place

We introduce the space of kinematically and physically (impenetration) admissible configurations

$$\mathcal{C}_c = \{(\boldsymbol{r}, \boldsymbol{R}) \in H^1\left([0, L]; \mathbb{R}^3 \times SO(3)\right), \boldsymbol{r}(0) = \boldsymbol{r}(L), \boldsymbol{R} \text{ satisfies boundary conditions, and } d_{\boldsymbol{r}}(s) \leq 0,\ \forall s \in [0, L]\}.$$

This configuration space is not linear but rather a differentiable manifold.

If $\boldsymbol{f}$ is independent of $\boldsymbol{r}$, then the problem of finding the equilibrium configurations leads to the following nonlinear minimization problem

$$\begin{cases} \text{Find } (\boldsymbol{r}, \boldsymbol{R}) \in \mathcal{C}_c \text{ such that} \\ \mathcal{J}(\boldsymbol{r}, \boldsymbol{R}) \leq \mathcal{J}(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{R}}), \quad \forall (\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{R}}) \in \mathcal{C}_c, \end{cases} \tag{4}$$

where $\mathcal{J}$ is the energy expression for the elastic rod in balance.

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