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# ON THE LEAST ACTION PRINCIPLE

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## ABSTRACT

Classical dynamics may be conventionally considered to be born about $1687$ with NEWTON's *Principia* and grew up to a well-established theory in the fundamental works on the subject by EULER, LAGRANGE, HAMILTON and JACOBI during the XVIII century and the first half of the XIX century. EULER's law refers to motions of arbitrary bodies in the eucliden space but still, in most modern textbooks, the presentation of the foundations of dynamics is developed in the spirit of rigid body dynamics and with reference to finite dimensional systems [3]. Extensions to continuous systems are illustrated in [2], [4], [1] by assuming that the configuration manifold is modeled on a BANACH space. Anyway these treatments essentially reproduce the formal structure of the dynamics of finite dimensional systems with suitable technical changes required by functional analysis.

The long controversy concerning the least action principle started with the ugly dispute in $1751$ between MAUPERTUIS and SAMUEL KÖNIG who claimed a plagiarism of a previous result communicated in $1707$ by LEIBNIZ in a letter to JACOB HERMANN. VOLTAIRE, D'ALEMBERT and EULER where also involved at the center of the dispute, with EULER defending MAUPERTUIS since KÖNIG was not capable to show the incriminating letter.

In [3], footnote on page $243$, ARNOLD says: *In almost all textbooks, even the best, this principle is presented so that it is impossible to understand (C. Jacobi, Lectures on Dynamics, 1842 - 1843). I do not choose to break with tradiction. A very interesting "proof" of Maupertuis principle is in Section 44 of the mechanics textbook of Landau and Lifšhits (Mechanics, Oxford, Pergamon, 1960).*

In [2], footnote on page $249$, ABRAHAM & MARSDEN write: *We thank M. Spivak for helping us to formulate this theorem correctly. The authors, like many others (we were happy to learn), were confused by the standard textbook statements*.

Anyway, despite these recent contributions, the least action principle is still formulated as a special result concerning systems undergoing conservative dynamical processes. We present here for the first time a formulation of MAUPERTUIS' principle as a general variational statement of the law of dynamics, with the aim of ending its long and laborious track along the last two centuries.

The analysis is developed in the context of continuum dynamics and accordingly the configuration-space is assumed to be a differentiable manifold modeled on a BANACH space. The standard form of the least action principle, as stated by MAUPERTUIS in $1744$ [5] and subsequently restated in more precise

terms by EULER, LAGRANGE and JACOBI, has been reproduced with no exception in the textbooks on dynamics [3]. It characterizes a trajectory as a stationarity path for the integral of the reduced action.

We show here that MAUPERTUIS' least action principle can be formulated in fairly general terms, valid for any dynamical system and including time-dependent energy functionals and non-potential forces. In this general form it states that, in the velocity phase-space, a trajectory of the system governed by a time-dependent energy functional and subject to an arbitrary system of forces, is a stationarity path for the line integral of the canonical one-form. Virtual velocity fields are constrained to be such that the virtual rate of change of the energy functional is equal to the virtual power performed by the force system.

The least action principle provides the geometry of the trajectory but not the time law according to which it is travelled by the dynamical system, since the EULER condition is homogeneous in the trajectory speed. The time schedule along the trajectory can however be detected by the energy variation law.

In this new form MAUPERTUIS least action principle is a general variational principle of dynamics.

The equivalence between MAUPERTUIS' principle and HAMILTON's principle is given a simple and clear mathematical proof by means of LAGRANGE's multiplier method. To this end a general form of the method is enunciated and proved on the basis of BANACH's closed range theorem in functional analysis.

Another important feature of the new formulation of the action principle is that no fixed end conditions are imposed in performing the virtual variations of the trajectory. The formulation of an action principle with no end conditions appears to be new and is based on a geometric approach. The stationarity condition is expressed by imposing that the variation of the signed-lenght of the extremal path due to virtual displacements of the path in the phase-space must be equal to the variation due to the virtual displacements of the path end-points. In deriving the differential condition of stationarity the REYNOLDS transport theorem, the AMPÈRE-HANKEL-KELVIN transform, usually dubbed STOKES's formula, and its expression in terms of differential forms due to POINCARÉ, the CARTAN's magic formula and the PALAIS' formula for the exterior derivative of a differential one-form, are the playmates [6].

The new general statement of the action principle provides new computational methodologies in dynamics and in related fields of mathematical physics.

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