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A multilevel Galerkin boundary element method

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ABSTRACT

The boundary element method (BEM) is a well known numerical method in the analysis of many physical problems. The traditional BEM, however, often leads to a dense matrix A_h, thus setting it up and performing a matrix-vector multiplication is an $\mathcal{O}(N^2)$ operation, where N is the degrees of freedom. Different schemes have been developed to reduce the complexity from $\mathcal{O}(N^2)$ to an almost linear complexity $\mathcal{O}(N \log^a N)_{a \geq 0}$, e.g., the panel clustering technique [3], multipole expansion method [2], hierarchical matrices ($\mathcal{H}$-matrices) [4] and the wavelet compression method [1].

In this paper, first we propose a procedure to coarsen the standard boundary element space to produce a sequence of lower-dimensional subspaces of it. Then the new coarsening bases are used to construct a data-sparse approximation $A_{\mathcal{H}}$ of the boundary element matrix A_h arising from the discretization of boundary integral equations by Galerkin method. This method has $\mathcal{O}(N)$ complexity and is suitable for solving problems with complicated surfaces in the three dimensional space. It can be recognized as a generalization of the wavelet Galerkin BEM in [1]. The basic ideas are as follows.

1. Multilevel bases

We generate a sequence of subspaces $V_{2 \leq j \leq J}$ of boundary element space X_h, i.e., $V_2 \subset V_3 \subset \cdots \subset V_J = X_h$ based on the hierarchical subdivision of the boundary Γ, see e.g., [1,2]. Let $\mathscr{C}_j$ be the set of non-empty cubes on level j obtained by the space subdivision. Due to the continuous subdivision, certain father-son relations is assigned to cubes in $\mathscr{C}_j$ and $\mathscr{C}_{j+1}$.

The key to our construction is the coarsening (transform) matrices Q_c for cube $c \in \mathscr{C}_{2 \leq j \leq J}$. Q_c are obtained via the singular value decomposition (SVD) of the moment matrices M_c (see [1] for definition). If Q_c are obtained, let $\Phi_{\text{sons}(c)}$ consists of all the basis functions in the sons of c, then the basis functions in c is obtained by

$$\Phi_c = Q_c^\top \Phi_{\text{sons}(c)}, \quad c \in \mathscr{C}_j,\ 2 \leq j \leq J.$$

Note that when $c \in \mathscr{C}_J$, $\Phi_{\text{sons}(c)}$ consists of boundary element basis functions in c. By performing the above transformation recursively from J to 2, we obtain a sequence of multilevel coarsening boundary element bases. The complexity of this construction is $\mathcal{O}(N)$.

2. Matrix approximation

The multilevel approximation method combines many aspects of the current fast BEMs, e.g., [1-4]. Matrix A_h is decomposed according to the neighbors of cubes in every level; that is

$$A_h = A_{\rm near} + \sum_{j=2}^{J} A_j.$$

where $A_{\rm near}$ consists of the interactions of boundary element basis functions in the neighbors and is computed using quadratures as in the traditional BEM. A_j consists of the interactions of boundary element basis functions in the interaction lists (see [2]) in level $2 \le j \le J$.

Let $A^{\phi}_{c,c'}$ consists of interactions of coarsening basis functions in c and c', i.e.,

$$A^{\phi}_{c,c'} = \langle \Phi_{c,{\rm L}}, \mathcal{K}\Phi^{\top}_{c',{\rm R}} \rangle,$$

where "R" and "L" in the subscripts indicate the right and left basis [1], and A^{ϕ}_{j} be the matrices obtained by replacing the blocks corresponding to c and c' in A_j by $A^{\phi}_{c,c'}$. Then we show that A_j can be approximated as

$$A_j \approx \tilde{A}_j := \big(Q_{J,{\rm L}} \cdots Q_{j,{\rm L}}\big) A^{\phi}_{j} \big(Q^{\top}_{j,{\rm R}} \cdots Q^{\top}_{J,{\rm R}}\big),$$

where Q_j are diagonal block matrices consists of transform matrices Q_c of all j-level cubes. Thus, we achieve our multilevel approximation of A_h

$$A_h \approx A_{\mathcal{H}} := A_{\rm near} + A_{\rm far}.$$

where

$$A_{\rm far} = \sum_{j=2}^{J} \tilde{A}_j = Q_{J,{\rm L}}\Big(A^{\phi}_{J} + Q_{J-1,{\rm L}}\Big(\cdots + Q_{3,{\rm L}}\big(A^{\phi}_{3} + Q_{2,{\rm L}}A^{\phi}_{2}Q^{\top}_{2,{\rm R}}\big)Q^{\top}_{3,{\rm R}}\cdots\Big)Q^{\top}_{J-1,{\rm R}}\Big)Q^{\top}_{J,{\rm R}}.$$

The approximate matrix $A_{\mathcal{H}}$ obtained in this way is data-sparse (only few data are needed for its representation) and typically has a hierarchical structure (see [4]). Thus the computational operations and memory requirement of $A_{\mathcal{H}}$ is $\mathcal{O}(N)$. An matrix-vector multiplication scheme with $\mathcal{O}(N)$ operations is also presented according to the definition of $A_{\rm far}$. The validation of the method is verified by numerical examples.

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