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Coupling Discontinuous Galerkin Discretizations using Mortar Finite Elements for Advection-Diffusion-Reaction Problems

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ABSTRACT

Discontinuous Galerkin (DG) methods employ discontinuous piecewise polynomials to approximate the solutions of differential equations with boundary conditions and interelement continuity weakly imposed through bilinear forms. Examples of these schemes include the Bassy-Rebay method, the Local Discontinuous Galerkin (LDG) methods, the Oden-Babuška-Baumann (OBB-DG) method and interior penalty Galerkin methods. Even though DG solvers can be expensive due to the number of unknowns, DG methods are of particular interest for multiscale problems with several appealing properties: They are element-wise mass conservative; they support local approximations of high order; they are robust and nonoscillatory in the presence of high gradients; they are implementable on unstructured and even non-matching grids; and with appropriate meshing, they are capable of delivering exponential rates of convergence.

On the other hand, non-overlapping domain decomposition is a useful approach for spatial coupling/decoupling. A subsurface flow example is the multiblock mortar mixed finite element (MFE) method described in [1]. There, the governing equations hold locally on the subdomains and physically driven matching conditions are imposed on block interfaces in a numerically stable and accurate way using mortar finite element spaces. References on the mortar approach for other discretizations include [5,4,3] for conforming Galerkin and [7] for finite volume elements. Couplings of DG and MFE methods have been also studied in the literature. A method for coupling LDG and MFE is developed in [6] by choosing appropriate numerical fluxes on interface edges. In [2], a multiscale MFE method was introduced for modeling Darcy flow. There, the continuity of the flux is imposed via a mortar finite element space on a coarse grid scale, while the equations in the coarse elements (or subdomains) are discretized on a fine grid scale. Optimal fine scale convergence is obtained by an appropriate choice of mortar grid and polynomial degree of approximation. In [10], multiscale mortar MFE-DG/DG-DG coupling methods were developed for pure diffusion problems. In [9], a multiscale mortar MFE method was also developed for nonlinear parabolic problems.

In this talk [8], we consider a general advection-diffusion-reaction problem. This class of equation includes second-order elliptic and parabolic equations, advection-reaction equations, as well as problems of mixed hyperbolic-elliptic-parabolic type. We study multiscale mortar DG-DG coupling methods based on four different DG formulations, the OBB-DG, the nonsymmetric interior penalty Galerkin (NIPG), the symmetric interior penalty Galerkin (SIPG), and the incomplete interior penalty Galerkin (IIPG). A Robin-type matching condition which involves a flux jump term and a penalized solution jump term, is imposed. The matching condition guarantees weak continuity of the total flux and the solution on the interface. The mortar variable is used as a Lagrange multiplier on the interface. The subdomain grids need not match and the mortar grid may be much coarser, giving a two-scale method. Convergence results in terms of the fine subdomain scale h and the coarse mortar scale H are established. We allow the diffusion coefficient to be degenerate and derive uniform convergence estimates. We show that DG-DG coupling using mortars preserves the same order of convergence rates of the DG for the advection-reaction equation with a proper choice of the penalty parameter on the interface. This optimal order of convergence rate is obtained without restriction on the choice of mortar grid and the polynomial degree of approximation in contrast to the case of the multiscale mortar MFE. A non-overlapping parallel domain decomposition algorithm reduces the coupled system to an interface mortar problem. The properties of the interface operator are discussed.

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