8th. World Congress on Computational Mechanics (WCCM8)
5th. European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30 – July 5, 2008
Venice, Italy

NUMERICAL IMPLEMENTATION OF A NEW CLASS OF VARIATIONAL PLASTICITY MODELS FOR GEOMATERIALS

***K. Krabbenhoft[1]**

[1] Centre for Geotechnical and Materials Modelling
University of Newcastle, NSW, Australia
E-mail: kristian.krabbenhoft@newcastle.edu.au
URL: http://livesite.newcastle.edu.au/cgmm

Key Words: *Plasticity, geomaterials, variational principles, implicit integration.*

ABSTRACT

Recently, a new class of elastoplasticity models for geomaterials has been proposed [1,2]. In contrast to existing modeling paradigms, the new framework is variational in nature. Essentially, the framework makes use of a generalized form of von Mises's principle of maximum plastic dissipation given by

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{\sigma}^{\mathsf{T}}\dot{\boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^{\mathsf{T}}\mathbf{C}\dot{\boldsymbol{\sigma}} - \boldsymbol{\kappa}^{\mathsf{T}}\mathbf{G}\dot{\boldsymbol{\kappa}} - \boldsymbol{\kappa}^{\mathsf{T}}\mathbf{V}\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^{\mathsf{T}}\mathbf{V}^{\mathsf{T}}\dot{\boldsymbol{\kappa}} \\ &\text{subject to} \quad F(\boldsymbol{\sigma},\boldsymbol{\kappa}) \le 0 \end{aligned} \tag{1}$$

where $\hat{\mathbf{C}}$, $\mathbf{G}$, and $\mathbf{V}$ are constitutive moduli. The associated Karush-Kuhn-Tucker optimality conditions are given by

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}} &= \hat{\mathbf{C}}\dot{\boldsymbol{\sigma}} + \mathbf{V}^{\mathsf{T}}\dot{\boldsymbol{\kappa}} + \dot{\lambda}\nabla_{\sigma}F(\boldsymbol{\sigma},\boldsymbol{\kappa}) \\ \mathbf{0} &= \mathbf{V}\dot{\boldsymbol{\sigma}} + \mathbf{G}\dot{\boldsymbol{\kappa}} + \dot{\lambda}\nabla_{\kappa}F(\boldsymbol{\sigma},\boldsymbol{\kappa}) \\ &F(\boldsymbol{\sigma},\boldsymbol{\kappa}) \le 0,\ \dot{\lambda}F(\boldsymbol{\sigma},\boldsymbol{\kappa}) = 0,\ \dot{\lambda} \ge 0 \end{aligned} \tag{2}$$

These equations define a class of models capable of capturing the typical features of geomaterials. In particular, any "effective flow rule" can be accommodated by choosing the constitutive moduli appropriately. One such choice [1,2] leads to the following governing equations:

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}} &= \mathbf{C}\dot{\boldsymbol{\sigma}} + \dot{\lambda}[\nabla_{\sigma}F(\boldsymbol{\sigma},\boldsymbol{\kappa}) + \mathbf{S}^{\mathsf{T}}\nabla_{\kappa}F(\boldsymbol{\sigma},\boldsymbol{\kappa})] \\ \dot{\boldsymbol{\kappa}} &= \mathbf{S}\dot{\boldsymbol{\sigma}} + \dot{\lambda}\mathbf{h} \\ &F(\boldsymbol{\sigma},\boldsymbol{\kappa}) \le 0,\ \dot{\lambda}F(\boldsymbol{\sigma},\boldsymbol{\kappa}) = 0,\ \dot{\lambda} \ge 0 \end{aligned} \tag{3}$$

where $\mathbf{C}$, $\mathbf{S}$, and $\mathbf{h}$ are new (physical) constitutive moduli. Thus, the effective flow rule is governed by the modulus $\mathbf{S}$ which is not present in conventional elastoplastic models.

Straightforward manipulations of the governing equations (3) lead to the following incremental relation between stresses and strains:

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}^{ep}\dot{\boldsymbol{\sigma}}, \quad \mathbf{C}^{ep} = \mathbf{C} + \frac{1}{H}\tilde{\mathbf{a}}\tilde{\mathbf{a}}^{\mathsf{T}} \tag{4}$$

where

$$\tilde{\mathbf{a}} = \nabla_{\sigma}F(\boldsymbol{\sigma},\boldsymbol{\kappa}) + \mathbf{S}^{\mathsf{T}}\nabla_{\kappa}F(\boldsymbol{\sigma},\boldsymbol{\kappa}),\ \ H = -\mathbf{h}^{\mathsf{T}}\nabla_{\kappa}F(\boldsymbol{\sigma},\boldsymbol{\kappa}) \tag{5}$$

The “continuum” elastoplastic tangent modulus is thus always symmetric.

The governing equations (2) are easily extended to finite-size increments. Using a backward Euler approximation we have

$$\begin{aligned} \Delta\boldsymbol{\varepsilon} &= \hat{\mathbf{C}}(\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n) + \mathbf{V}^{\mathsf{T}}(\boldsymbol{\kappa}_{n+1} - \boldsymbol{\kappa}_n) + \lambda_{n+1}\nabla_{\sigma}F(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\kappa}_{n+1}) \\ \mathbf{0} &= \mathbf{V}(\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n) + \mathbf{G}(\boldsymbol{\kappa}_{n+1} - \boldsymbol{\kappa}_n) + \lambda_{n+1}\nabla_{\kappa}F(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\kappa}_{n+1}) \\ & F(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\kappa}_{n+1}) \le 0,\ \dot{\lambda}_{n+1}F(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\kappa}_{n+1}) = 0,\ \dot{\lambda} \ge 0 \end{aligned} \tag{6}$$

where $\Delta\boldsymbol{\varepsilon}$ is considered given. These equations follow from a maximization principle given by

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{\chi}_{n+1}^{\mathsf{T}}\Delta\boldsymbol{\alpha} - \tfrac{1}{2}(\boldsymbol{\chi}_{n+1} - \boldsymbol{\chi}_{\text{tr}})^{\mathsf{T}}\mathbf{L}(\boldsymbol{\chi}_{n+1} - \boldsymbol{\chi}_{\text{tr}}) \\ &\text{subject to} \quad F(\boldsymbol{\chi}_{n+1}) \le 0 \end{aligned} \tag{7}$$

or, equivalently, by

$$\begin{aligned} &\text{maximize} \quad -\tfrac{1}{2}(\boldsymbol{\chi}_{n+1} - \boldsymbol{\chi}_{\text{tr}})^{\mathsf{T}}\mathbf{L}(\boldsymbol{\chi}_{n+1} - \boldsymbol{\chi}_{\text{tr}}) \\ &\text{subject to} \quad F(\boldsymbol{\chi}_{n+1}) \le 0 \end{aligned} \tag{8}$$

where

$$\boldsymbol{\chi} = \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\kappa} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\chi}_{\text{tr}} = \begin{pmatrix} \boldsymbol{\sigma}_n + \mathbf{C}^{-1}\Delta\boldsymbol{\varepsilon} \\ \boldsymbol{\kappa}_n \end{pmatrix}, \quad \mathbf{L} = \begin{bmatrix} \hat{\mathbf{C}} & \mathbf{V}^{\mathsf{T}} \\ \mathbf{V} & \mathbf{G} \end{bmatrix} \tag{9}$$

This is a standard closest-point projection problem [3] which can be solved either “directly”, by application of Newton’s method to (6), or by means of more elaborate methods from the optimization literature [4].

The finite-step material-point principle (7) can be extended to the spatial domain of interest by the following Hellinger-Reissner type principle:

$$\begin{aligned} &\min_{\mathbf{u}_{n+1}} \max_{\boldsymbol{\chi}_{n+1}} \quad \int_V \Big\{ \boldsymbol{\chi}_{n+1}^{\mathsf{T}}\Delta\boldsymbol{\alpha} - \tfrac{1}{2}\Delta\boldsymbol{\chi}_{n+1}^{\mathsf{T}}\mathbf{L}\Delta\boldsymbol{\chi}_{n+1} + \Delta\mathbf{u}_{n+1}^{\mathsf{T}}(\nabla\boldsymbol{\sigma}_{n+1} - \mathbf{b}) \Big\}\,\mathrm{d}V - \int_S \Delta\mathbf{u}_{n+1}^{\mathsf{T}}\mathbf{t}\,\mathrm{d}S \\ &\text{subject to} \quad F(\boldsymbol{\chi}_{n+1}) \le 0 \end{aligned}$$

where $\Delta\mathbf{u}_{n+1} = \mathbf{u}_{n+1} - \mathbf{u}_n$ are the displacements, $\mathbf{b}$ are the body forces, and $\mathbf{t}$ are the tractions. This problem may be discretized by finite elements and subsequently solved using the conventional two-stage procedure of Simo [3] or by means of the optimization inspired scheme proposed by Krabbenhoft *et al.* [5]. In the presentation both approaches will be considered.

REFERENCES

[1] K. Krabbenhoft *et al.*, “Variational plasticity models for geomaterials”, *Proc. COMPLAS IX*, Barcelona, (2007).

[2] K. Krabbenhoft “A variational theory of plasticity for frictional materials”, Research Report, University of Newcastle, NSW (2008).

[3] J.C. Simo, *Topics on the Numerical Analysis and Simulation of Plasticity*, in: Handbook of Numerical Analysis, Vol. III, Elsevier, (1995).

[4] R.J. Vanderbei, *Linear Programming Foundations and Extensions*, Springer.

[5] K. Krabbenhoft, A.V. Lyamin, S.W. Sloan, P. Wriggers., “An interior-point algorithm for elastoplasticity”, *Int. J. Numer. Meth. Eng.*, **69**, 592-626, (2007).