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Stability analysis and numerical implementation of non-local damage models via a global variational approach

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ABSTRACT

Brittle fracture can be formulated as a variational problem [1]. In the case of linear elastic solids with Griffith surface energy, the crack evolution is obtained by minimizing the energy functional

$$\mathcal{E}(u,\Gamma) = \int_{\Omega\setminus\Gamma} \frac{1}{2} A_0\, \epsilon(u)\cdot\epsilon(u)\, dx + G_c\, \mathcal{H}^{n-1}(\Gamma) \tag{1}$$

over all the admissible displacement fields u and fracture surfaces Γ, where $\epsilon(u)$ is the strain, A_0 the elasticity tensor, G_c the fracture toughness, and $\mathcal{H}^{n-1}$ the Haursdoff surface measure. The current numerical implementation of this variational approach is based on the approximation of the energy functional (1) with a regularized version, suitable for a finite element discretization [2]. Mechanically, this corresponds to the approximation of the brittle fracture problem with a special class of gradient damage models caracterized by a total energy of the form:

$$\mathcal{E}_\ell(u,\alpha) = \int_\Omega \frac{1}{2} A(\alpha)\, \varepsilon(u)\cdot\varepsilon(u)\, dx + G_c \int_\Omega \left(\frac{w(\alpha)}{\ell} + \ell\, \nabla\alpha\cdot\nabla\alpha \right) dx \tag{2}$$

where ℓ is a characteristic length, $A(\alpha)$ represents the elasticity tensor at the damage state α, and $w(\alpha)$ can be interpreted as the density of the dissipated energy during a homogeneous damage process. Under some conditions for the form $A(\alpha)$ and $w(\alpha)$, Γ-convergence results [3] garantee the convergence of such damage models to the brittle fracture problem when $\ell \to 0$. This means that global minima of (2) converges toward global minima of (1). From the physical point of view, the relevance of the global minima is questionable. The search of local minima of the energy seems more pertinent to follow fracture (damage) evolution from a given state. In this context, even if Γ-convergence results do not apply, the damage energy (2) remains a well-founded approximation of the Griffith energy. However, different choices of the constitutive relations lead to different behaviors with respect to local minima. Hence, the question of how to select the constitutive relations $A(\alpha)$ and $w(\alpha)$ arises. The phenomenological pertinence of different constitutive assumptions may be addresses by the analysis of

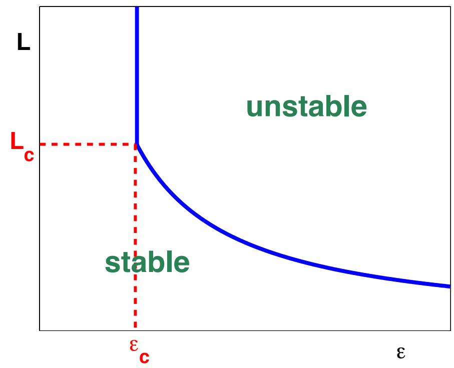

Figure 1: Critical bar length for the stability of homogeneously damaged states as a function of the mean deformation (traction test).

the stability property of the corresponding damage models.

The present study analyzes damage models of the class (2) with three different constitutive relations of the type:

$$A(\alpha) \quad = \quad (1-\alpha)^p, \qquad w(\alpha) = c\,\alpha^q, \qquad \alpha \in [0,1] \tag{3}$$

The analytical part of this work focuses on the investigation of the displacement-controlled traction of a bar [4] within a 1D setting. We report analytical results about (i) the properties of the strain-stress relationships, (ii) the limit of stability of the purely elastic solution, (iii) the range of stability of homogeneously damaged states (see e.g. Figure 1). Here, stable states are characterized as unilateral local minima of the energy functional (2), the unilateral condition following by the irreversibility of the damage evolution. These models are then implemented in a finite element code able to address 2D problems (plane elasticity). In the present variational framework, solutions are obtained by the numerical minimization of the discretized version of (2). In particular, we adapt the algorithm used in [2], which is based on alternate minimization of elasticity problems at fixed damage field and of damage problems at fixed displacement. Each damage problem appears as a quadratic bound constrained minimization, with at least a lower bound corresponding to the irreversibility condition on the damage field. The numerical finite element results are compared to the analytical findings for the traction test. This allows us to validate the finite element solution and investigate the capabilities of the alternate minimization algorithm in selecting stable evolutions. Finally, numerical results for more complex damage-fracture examples are reported.

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