8th. World Congress on Computational Mechanics (WCCM8)
5th. European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30 – July 5, 2008
Venice, Italy

GEOMETRICALLY CONSISTENT FORMULATIONS FOR CONSTRAINED SYSTEM DYNAMICS

Marco Borri[1], *Lorenzo Trainelli[2]

[1] Politecnico di Milano, Aerospace Eng. Dept.
Via La Masa, 34 – 20156 Milano
marco.borri@polimi.it
http://www.aero.polimi.it/~borri/

[2] Politecnico di Milano, Aerospace Eng. Dept.
Via La Masa, 34 – 20156 Milano
lorenzo.trainelli@polimi.it
http://www.aero.polimi.it/~trainell/

Key Words: *Differential-algebraic equations, Constrained dynamical systems, Constraint stabilization, Hidden constraints, Index reduction, Embedded projection.*

ABSTRACT

The mathematical formulation of the dynamics of constrained mechanical systems typically relies on systems of differential-algebraic equations (DAEs) where the differential part represents kinematic evolution equations and dynamic balance equations, while the algebraic part represents geometrical constraints. It is well known that the DAE nature of the governing equations often gives rise to difficulties in their numerical solution. In this regard, an important role is played by the inconsistencies between explicitly appended constraints and their time-integrals or time-derivatives (the so-called 'hidden' constraints).

In fact, explicit and 'hidden' constraints are consequential at the exact mathematical level, standard time integration approaches methods typically do not preserve this property at the numerical level. As a consequence, one may face serious convergence difficulties, accuracy loss for the algebraic variables, and general stability problems. On the other hand, we know that – when possible – a minimal co-ordinate set approach, which leads to a set of ordinary differential equations (ODEs), does not suffer the troubles mentioned above.

Therefore, we consider as a goal the reformulation of the problem that keeps the generality and flexibility of the redundant co-ordinate set approach (*i.e.* the DAE setting) while preserving the 'geometric' quality of the minimal-set approach. This quality is the same as the exact solution since it holds (a) at all relevant levels of differentiation, and (b) in a 'pointwise' manner, *i.e.* exactly at the time step boundaries, without approximations/averaging over the time step.

In the development towards this goal, we review some general formulations for constrained system dynamics. We consider holonomic as well as non-holonomic constraints, showing the relationships between the classical Lagrange multiplier framework, the "$\dot{\mu}$ –*Method*" conceived some 20 years ago, and the recent "*Embedded Projection Method*" (EPM). The analysis is carried out looking at differential and variational implications, in both the Lagrangian and Hamiltonian frameworks.

It is the Hamiltonian viewpoint, however, that reveals most helpful in understanding

how to take into account the intimate coupling of the algebraic and differential parts of the governing equations. This is done, in the EPM, by reformulating the problem in terms of *fully unconstrained* state variables in a modified phase space. In fact, the EPM can be seen as a general index reduction procedure from arbitrarily high DAE index to one in which constraints are not imposed on position-type variables only, since the corresponding restrictions are *independently* placed on velocity-type variables. These constraints, together with their time-derivatives, are used to define modified state variables which, under all respects, are free to evolve in their phase space.

As a result, an index-1 DAE set is obtained, *i.e.* an ODE set complemented with algebraic equations that do not act as constraints, but as implicit definitions of internal variables. This automatically results in greater accuracy and stability of the numerical solution: constraints are exactly satisfied at all relevant levels (position, velocity, acceleration) by construction, and nominal accuracy is fully recovered for all involved quantities, irrespective of the chosen numerical integrator.

In the discussion, it appears that the DAE index appears as a 'measure' of lack of information. In this regard, the EPM requires additional knowledge with respect to traditional approaches, but this additional knowledge does not constitute a burden in itself, since it is simply given by the constraint derivatives. The higher complexity of the procedure with respect to traditional approaches is paid back by a higher global quality of the numerical solution, together with a much improved numerical behaviour. Some numerical examples are included to illustrate the results predicted by the theory.

In conclusion, the EPM provides a rationale for interpreting constraints in a more complete framework than with more conventional approaches, and particularly justifies clever, albeit heuristic, recipes such as the well known *"GGL-method"*. Furthermore, we conjecture that a fully consistent Hamiltonian variational formulation corresponding to the EPM holds, as seen with the closely related *"$\dot{\mu}$ –Method"*.

REFERENCES

[1] M. Borri, P. Mantegazza, "Finite time element approximation of dynamics of nonholonomic systems", *Mod. Sim. Contr.*, Vol. **12**, pp. 7–18, (1987).

[2] M. Borri, C.L. Bottasso, P. Mantegazza, "A modified phase space formulation for constrained mechanical systems – Differential Approach", *Eur. J. Mech., A/Solids*, Vol. **11**, pp. 701–727, (1992).

[3] M. Borri, L. Trainelli, A. Croce, "The Embedded Projection Method: A General Index Reduction Procedure for Constrained System Dynamics", *Comp. Meth. Appl. Mech. Engrng.*, Vol. **195**, pp. 6974–6992, (2006).

[4] K.E. Brenan, S.L. Campbell, L.R. Petzold, *Numerical solution of initial-value problems in differential-algebraic equations*, Elsevier Science Publishing Co., New York, 1989.

[5] C.W. Gear, "Differential-algebraic equation index transformations", *SIAM J. Sci. Stat. Comput.*, Vol. **9**, pp. 40–47, (1988).