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# High-order Multistep Asynchronous Splitting Integrators (MASI)

**Minyong Shin[1] and *Matthew West[2]**

[1] Aeronautics and Astronautics
Stanford University
Stanford, CA 94305, USA
`mshin@stanford.edu`

[2] Mechanical Science and Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
`mwest@uiuc.edu`

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## ABSTRACT

We introduce a new class of time integration methods designed for efficient simulation of multiscale ODEs and PDEs. These methods, known as Multistep Asynchronous Splitting Integrators (MASI), are an extension of Asynchronous Splitting Methods (ASM) [3] and Asynchronous Variational Integrators (AVI) [2]. MASI methods generalize classical multistep integration methods [1] to split systems, allowing different timesteps to be used for each component of the system. For multiscale systems with different characteristic timescales for each component this allows more efficient integration than a classical synchronous method. The additional cost of the presented method is some degree of book-keeping and the per-step inverse of a matrix of size equal to the order of the method.

**Asynchronous time integration.** We consider ODEs (often spatially discretized PDEs) with a splitting of the form:

$$\dot{x}(t) = f(x(t)) = \sum_{i=1}^{M} f_i(x(t)). \tag{1}$$

An asynchronous time integration method takes different timesteps for each component $i$ of the vector field, and these timesteps happen independently of each other. At any stage we assume that the least-advanced component of the vector field is timestepped next. This defines an ordering of operations that we index by $k$.

Let $h_k$ be the timestep used at step $k$, and let $i_k$ be the index of the vector field component which is to be advanced. We can now define the cumulative time for each component $i$ to be $t_k^i$ and the global minimum time to be $t_k^{\min}$ given by:

$$t_k^i = \sum_{j=1}^{k} \delta_{i_j}^i h_j \tag{2}$$

$$t_k^{\min} = \min_{i=1,\dots,M} t_k^i. \tag{3}$$

These quantities are illustrated in figure 1.

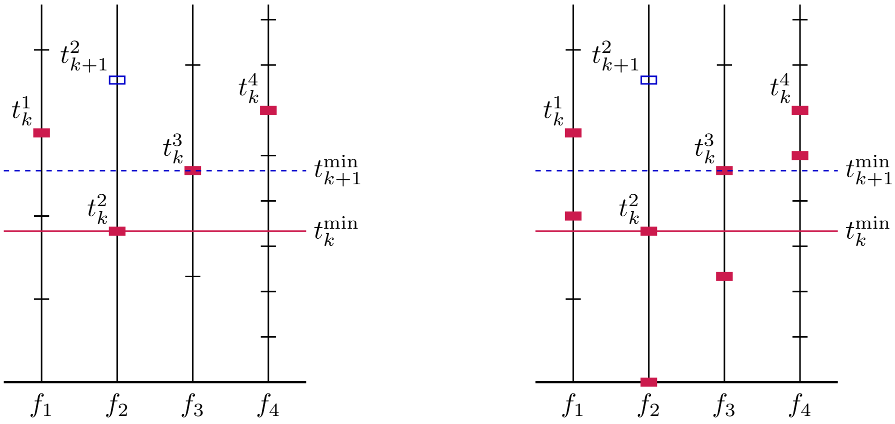


Figure 1: A schematic representation of an Asynchronous Splitting Integrator (ASI) at left and a Multistep Asynchronous Splitting Integrator (MASI) at right with two previous values stored.

**Multistep Asynchronous Splitting Integrators (MASI).** For an ODE decomposed as (1) we define a $p$-step linear multistep asynchronous splitting integrator to be an update of the form:

$$x_{k+1} + \sum_{\ell=p-n+1}^{p} \sum_{i=1}^{M} \alpha_{i,\ell} x_{i,\ell} = \sum_{\ell=p-n+1}^{p} \sum_{i=1}^{M} \beta_{i,\ell} f_i(x_{i,\ell}). \tag{4}$$

where $x_{i,\ell}$ is a previous system state numbered per-component so that $x_{i,p}$ is the most recent step of the $i$-th component. Here $\alpha$ and $\beta$ are per-step coefficients chosen to satisfy order and stability properties, while $x_k$ is the approximate state of the system at update $k$.

**A simple stable MASI.** A particular simple example of a MASI can be formed by choosing a last-point-only scheme (not defined here) for $\alpha$ to ensure stability, and then choosing $\beta$ to satisfy the order conditions. Such a scheme stores $n$ previous system states, is $n$-th order, and requires the inversion of a $n \times n$ matrix at each timestep. We show that this method is stable, consistent and convergent.

We provide numerical evidence to show that for multiscale systems with significant scale separation high-order MASI methods are cheaper on a cost versus error basis than either low-order asynchronous methods or high-order synchronous multistep or Runge-Kutta methods. The improvement achievable by MASI methods increases with increasing scale separation.

## REFERENCES


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