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GREEN'S FUNCTIONS AND FINITE ELEMENTS OR WHY STRESSES ARE RARELY ACCURATE

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ABSTRACT

To calculate displacements, stresses or support reactions, etc. in an FE-model means to apply a functional $J(u)$ to the FE-solution $J(u) = u(\boldsymbol{x})$, $J(u) = \sigma_{xx}(\boldsymbol{x})$. Technically each such functional can be represented by a function G_0

$$J(u) = \int_\Omega G_0(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}}$$

where G_0 is of course the Green's function. In FE-analysis the original Green's function G_0 gets replaced by its FE-projection G_0^h which means that in FE-analysis the output functionals $J(.)$ are replaced by functionals $J_h(.)$

$$J(u) = u(\boldsymbol{x}) = \int_\Omega G_0(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} \qquad \rightarrow \qquad J_h(u) = u_h(\boldsymbol{x}) = \int_\Omega G_0^h(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}}\,.$$

Note that this happens automatically! A FE-program calculates each and every value which appears on the screen by forming the L_2-scalar product between the associated Green's function and the right-hand side p. It is only that it substitutes for the exact Green's functions the FE-projection G_0^h. Now Green's function usually have an infinite energy and so we conclude that the output values of an FE program are based on the approximate solution of 'ill-posed problems', namely to find in $V_h \subset V$ (= the energy space) an element G_0^h which comes close in the sense of the energy metric to $G_0 \notin V$.

So FE-analysis is not only an exercise in numerical mathematics, but it also has to do with the choice of a *substitute mechanical model*, $G_0 \rightarrow G_0^h$. And the FE-approximation u_h of u on the chosen mesh is characterized by the statement that

$$J(u_h) = J_h(u) \qquad \text{for all functionals } J(.)\,.$$

That is we can establish the equivalence between u_h and u via the functionals $J(.)$. With regard to the mapped functionals, $J(.) \rightarrow J_h(.)$, the FE-solution u_h and u are identical, [1].

Obviously then the error $J(e) = J(u) - J(u_h)$ of the FE-solution

$$u(\boldsymbol{x}) - u_h(\boldsymbol{x}) = \int_\Omega (G_0(\boldsymbol{y}, \boldsymbol{x}) - G_0^h(\boldsymbol{y}, \boldsymbol{x}))\, p(\boldsymbol{y})\; d\Omega_{\boldsymbol{y}}$$

depends on how well the influence functions for the displacements, stresses, or support reactions can be approximated. This leads to 'goal oriented' adaptive refinement procedures [2]. The kernel for a displacement $u(\boldsymbol{x})$ is a very smooth function while the kernel for the first derivative $\sigma_{xx} \sim u_{,x}$ is a dipole with two opposing peaks. This is the reason why stresses are rarely exact: conforming functions cannot approximate dislocations very well. So the kinematics of a mesh determines the accuracy of an FE solution

- mesh = kinematics = accuracy of influence functions = quality of results .

That is a structure (or any other similar problem domain in physics) consists of infinitely many mechanisms—all bolted and fixed so that the structure can carry the load. But if we release one mechanism, apply a unit rotation or dislocation, then this will induce a movement in the structure and the work done by the load on acting through this movement is equal to $M(x) \cdot 1$ or $V(x) \cdot 1$, is equal to the Dirac energy [1]. In FE analysis we hinder the movements of the structure and so the mechanism gets the wrong signal of how large the Dirac energy really is, and consequently $M_h(x) \neq M(x)$ and $V_h(x) \neq V(x)$.

But the Green's function also drive model adaptivity. A change in the stiffness of a frame element, $EI \rightarrow EI + \Delta EI$, leads to a change in any output functional $J(.)$

$$J(u_c) - J(u) = \int_0^l G_c\, p\, dy - \int_0^l G\, p\, dy\,.$$

By employing the 'weak' form of influence functions this change can be expressed 'locally'

$$J(u_c) - J(u) = -d(G_c, u) = -d(G_c - G, u) - d(G, u) \simeq -d(G, u) = \int_{x_a}^{x_b} \frac{\Delta EI}{EI} \frac{M_G\, M}{EI} dy$$

where $d(G_c, u)$ is a symmetric bilinear form which extends *only* over the element $[x_a, x_b]$ where the change occurs. So that also the influence of modifications can be estimated via the corresponding Green's functions. Sensitivity analysis makes use of Green's functions to calculate the derivatives of certain output functionals with respect to a change in the design variables, [3].

In nonlinear problems there are no Green's functions but the Green's functions at the linearization point can be used to steer an adaptive refinement process also in nonlinear problems and virtually the same technique is applied in sensitivity analysis of nonlinear problems, [4].

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