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DEVELOPMENT AND APPLICATIONS OF THE INVERSE OPERATORS IN CONTINUUM MECHANICS

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ABSTRACT

In this presentation the theory and specially the inverse of high order tensors are discussed. In our approach the special high order tensor, namely the *deviatorizable tensor* and the operator are determined as a tensor valued function of a deformation (strains). The deviatorizable tensor is a proper projector. As an application the deviatoric metal-plasticity with finite strains is researched [1]. Anisotropy of the material response is not precluded. When the large strains occur affected generally from large rotations many difficulties arises. One problematic task is the care of the objectivity, because the constitutive relations including stress rates have to take into account the rotation of the material. Then it is clear that the deviatoric and the inverse tensors or operators have to be satisfied these requirements too in addition to the existence and if possible, the uniqueness. Also to be invertible, the symmetry is restricted and a certain regularity or smoothness of a deformation is required. We will also show that the general deviatoric tensor is idempotent which property is necessary for a proper projector. The structure of the final presentation is as follows. First, the brief synopsis of the tensor or the multilinear algebra is given [2], [3], [4], [7]. For that only the essential and the needed topics are included and specially the invertibility of high order tensors is discussed. After that the generalized deviatoric tensor and the operator are developed. Finally, as a most essential topic of the paper, the inverse of the fourth order tensor is defined. By authors knowledge the general inverse tensor is verified with the certain assumptions in the advanced multilinear algebra but it is never launched or applied in general and closed form to the continuum mechanics. In our approach it is expressed using the absolute or tensorial notation to emphasize its tensorial nature. Indeed, it is independent of the selection of co-ordinate system. Also, as an mathematical operator or a projector it is very general and thus widely applicable in continuum or micro-mechanics. In general the deviatoric mapping or the *deviatorizer* and the corresponding operator can be defined as

$$\mathcal{P}^D : Lin \rightarrow Lin^D \subset Lin, \quad \mathcal{P}^D : \mathcal{P}^D = \mathcal{P}^D \, (idempotent), \quad \mathtt{tr}(\mathcal{P}^D) = 0, \tag{1a}$$

$$DEV[\cdot] : Lin \rightarrow Lin^D \subset Lin, \quad \mathtt{tr}\big(DEV[\cdot]\big) = 0 \tag{1b}$$

respectively, where Lin, Lin^D denote the space of linear and linear deviatoric mappings respectively and $\mathtt{tr}$ is the trace of tensor. Above a tensor is regarded as a bijective or an isomorphic mapping [3], [6]

and in the tensor algebra it is regarded to be of an even order, primarily a two or a fourth order tensor [4], [5]. Often those mappings are required to be invertible when their sub-symmetry is restricted or they belong to the special symmetry group $Inv \subset Lin$. Alternatively, let us assume that

$$\exists\, \Xi : sLin \to Inv \quad \text{and} \quad \Xi :\, Inv \to sLin, \tag{2}$$

where $sLin \subset Lin$ is the space of linear mappings with certain sub-symmetries. I.e. Ξ is a general *transposition projection tensor* for which

$$\begin{aligned} \Xi \circ (\Xi \circ {}'\mathcal{C}') = \Xi \circ {}'({}'\mathcal{C}')' = \Xi \circ \mathcal{C} = {}'\mathcal{C}' \in sLin, \\ (\Xi \circ \mathcal{P}^D) \circ {}'\mathcal{C}' = \Xi \circ (\mathcal{P}^D \circ {}'\mathcal{C}') \end{aligned}$$

or it satisfies the associative property with $\mathcal{P}^D$. Note, that this mapping is often available in continuum mechanics and it is associated to the *principle of virtual power* [8] after the linearization as

$$\mathcal{P}^{int} :=' \mathcal{C}' :: \dot{\mathbf{C}} \times \delta\mathbf{C} := \delta\mathbf{C} : \mathcal{C} : \dot{\mathbf{C}} \quad \forall \mathcal{C} \in Inv \subset Lin,\, \delta\mathbf{C}, \dot{\mathbf{C}} \in Sym \subset Lin,$$

where Sym is a space of symmetric linear mappings. If $\mathcal{P}^{int} > 0$, $\mathcal{C}$ is invertible almost everywhere. Thus, $\exists\, (\Xi \circ \mathcal{P}^D) : sLin \to Inv \subset Lin^D$. E.g.

$$\begin{aligned} (DEV[{}'\mathcal{C}'])^{-1} &:= \Xi \circ \big((\Xi \circ \mathcal{P}^D) \circ' \mathcal{C}'\big)^{-1} = \Xi :: \big((\Xi :: \mathcal{P}^D) : {}'\mathcal{C}'\big)^{-1} \\ &= \Xi :: \big(\Xi :: (\mathcal{P}^D : {}'\mathcal{C}')\big)^{-1} = \Xi :: \big(\Xi :: DEV[{}'\mathcal{C}']\big)^{-1} = {}'\big(({}'DEV[{}'\mathcal{C}']')^{-1}\big)', \end{aligned} \tag{3}$$

By author's experience, the development of the proper and theoretically correct deterministic models in applied continuum mechanics requires some systematic approach. As an example the deviatoric and the inverse tensors, developed from the basis of the tensor analysis seem to be beneficial specially when the consistent tangent tensors are developed for complex models including path-dependent and possibly anisotropic material behavior. That is, because by means of these closed-form operators the final constitutive relations reduce often to the remarkable simple but exact forms. This feature increases the accuracy and thus the convergence of a solution during the computation. As known, the consistent tangent tensor is preferred generally in the path-following algorithms in computational mechanics [1]. To illustrate the presentation, the obtained results are applied to the incompressible metal-plasticity with finite strains, where a certain deviatoric tangent tensor and its inverse are determined using Eqs (1)-(3).

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