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HIGH-ORDER DISCONTINUOUS GALERKIN METHODS FOR SOLVING CONSERVATION LAWS ON GENERAL 2D MANIFOLDS

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ABSTRACT

This work consists in a general method for solving partial differential equations on general 2D curved manifolds. We apply this technique to the particular case of the sphere with the aim to solve the shallow water equations on the Earth surface. In the framework of a new ocean model based on unstructured grids and finite elements, the discontinuous Galerkin method seems to be a good candidate since it allows the simple use of efficient techniques as high order polynomial function space, adaptivity and error estimation, efficient parallel computing, and exhibits superconvergence properties for the dissipation and dispersion errors.

Classical high order methods for solving the shallow water equations on the sphere consider a three-component velocity $\mathbf{v} = (u, v, w)$, each component being discretized as a scalar field, for a three-dimensional momentum equation. Those techniques do not garantee the velocity vectors to remain tangent to the manifold, and require then the use of explicit time schemes with a projection of the residual and the solution on the local tangent plan at every time step. We propose here a high order discontinuous Galerkin method considering vectorial test functions, taking into account the manifold curvature directly into the discrete operators. The manifold discretization is based on curved triangles with high order polynomial mappings.

Consider a surface $\mathcal{S}$ with a parametrization $\vec{x}(u,v)$ that maps a point $(u,v) \ \in \mathcal{R}^2$ from a reference element to $\vec{x} \ \in \mathcal{R}^3$ lying on $\mathcal{S}$ as depicted in Figure (1). For a sufficiently regular manifold $\mathcal{S}$, it is possible to define a tangent plane at any point on the surface by defining two tengential directions $(\vec{t_1}, \vec{t_2})$.

A tangential velocity field $\vec{v}(\vec{x}(u,v))$ can then be discretized to the polynomial order p as

$$\vec{v}(\vec{x}(u,v)) = \sum_{i=0}^{N} U_i L_i(\vec{x}(u,v)) \ \frac{\vec{t_1}(\vec{x})}{\parallel \vec{t_1}(\vec{x}) \parallel} + \sum_{i=0}^{N} V_i L_i(\vec{x}(u,v)) \ \frac{\vec{t_2}(\vec{x}(u,v))}{\parallel \vec{t_2}(\vec{x}) \parallel}$$

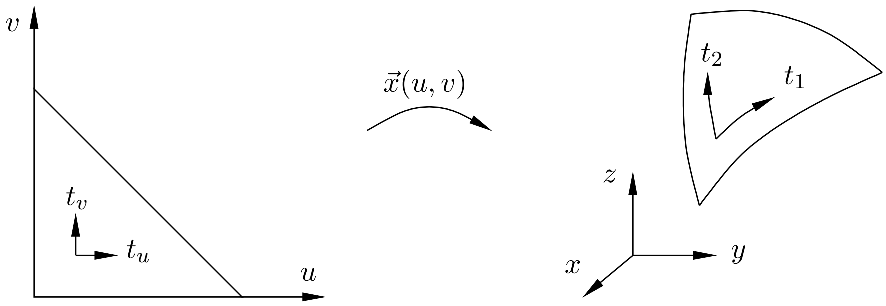

Figure 1: Mapping from the 2D reference element to a 2D curved manifold element defining two tangential vectors to the manifold at each point.

where L_i are the Lagrange polynomial basis functions in the reference framework, $\vec{t_1}$ and $\vec{t_2}$ are tangential velocity directions on the manifold at any point $\vec{x}(u, v)$, U_i and V_i are the velocity amplitudes at interpolation points in the directions $\vec{t_1}$ and $\vec{t_2}$ respectivelly and $N = (p+1)(p+2)/2$ is the number of nodal values associated to this order p. Normalizing the tangential directions is mandatory to be able to represent constant solutions on the curved surface.

A discontinuous Galerkin formulation is obtained by multiplying the momentum equation by a vectorial test function while keeping a scalar test function for the pressure in the continuity equation. Christofell symbols to take into account the manifold curvature are implicitly contained in the discrete operators. An implicit DIRK time scheme may then be applied since the discretization itself ensures tangent velocities. The method is validated by comparing the high-order discontinuous Galerkin results to classical benchmarks as the Williamson test cases used in atmospheric computations.

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