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Controllability method for time-harmonic acousto-elastic interaction

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ABSTRACT

We consider the use of a control algorithm to solve a time-harmonic acousto-elastic problem in the domain $\Omega \subset \mathbb{R}^2$, which is divided into the solid part Ω_s and the fluid part Ω_f by the interface Γ_i. Instead of solving directly the time-harmonic equation, we return to the corresponding time-dependent equation and look for time-periodic solution. The convergence is accelerated with a control technique by representing the original time-harmonic equation as an exact controllability problem [2, 5] for the time-dependent wave equation

$$\frac{1}{\rho_f(\mathbf{x})c(\mathbf{x})^2}\frac{\partial^2 p_f}{\partial t^2} - \nabla\cdot\left(\frac{1}{\rho_f(\mathbf{x})}\nabla p_f\right) = f, \qquad \text{in } \Omega_f \times [0,T], \tag{1}$$

$$p_f = 0, \qquad \text{on } \Gamma_{\text{0f}} \times [0,T], \tag{2}$$

$$\frac{1}{c(\mathbf{x})}\frac{\partial p_f}{\partial t} + \frac{\partial p_f}{\partial \mathbf{n}_f} = y_{\text{ext}}, \qquad \text{on } \Gamma_{\text{ef}} \times [0,T], \tag{3}$$

$$\rho_f(\mathbf{x})\frac{\partial^2 \mathbf{u}_s}{\partial t^2}\cdot\mathbf{n}_s - \frac{\partial p_f}{\partial \mathbf{n}_f} = 0, \qquad \text{on } \Gamma_{\text{i}} \times [0,T], \tag{4}$$

$$\rho_s(\mathbf{x})\frac{\partial^2 \mathbf{u}_s}{\partial t^2} - \nabla\cdot\sigma(\mathbf{u}_s) = \mathbf{f}, \qquad \text{in } \Omega_s \times [0,T], \tag{5}$$

$$\mathbf{u}_s = 0, \qquad \text{on } \Gamma_{\text{0s}} \times [0,T], \tag{6}$$

$$\rho_s(\mathbf{x})\mathbf{B}\frac{\partial \mathbf{u}_s}{\partial t} + \sigma(\mathbf{u}_s)\mathbf{n}_s = \mathbf{g}_{\text{ext}}, \qquad \text{on } \Gamma_{\text{es}} \times [0,T], \tag{7}$$

$$\sigma(\mathbf{u}_s)\mathbf{n}_s - p_f\mathbf{n}_f = 0, \qquad \text{on } \Gamma_{\text{i}} \times [0,T], \tag{8}$$

where f, y_{ext}, $\mathbf{f}$, and $\mathbf{g}_{\text{ext}}$ are the source terms. Length of the time interval is marked as T, p_f denotes the pressure, and $\mathbf{u}_s = (\mathbf{u}_{s1}, \mathbf{u}_{s2})^T$ is the displacement field. Coefficients $\rho_f(\mathbf{x}) > 0$ and $\rho_s(\mathbf{x}) > 0$ represent the densities of media in domains Ω_f and Ω_s, respectively, and $c(\mathbf{x}) > 0$ is the speed of sound in fluid domain. The stress tensor is expressed as $\sigma(\mathbf{u}_s) = \lambda(\nabla\cdot\mathbf{u}_s)\mathcal{I} + 2\mu\epsilon(\mathbf{u}_s)$ with the linearized strain tensor $\epsilon = \frac{1}{2}\left(\nabla\mathbf{u}_s + (\nabla\mathbf{u}_s)^T\right)$, the identity matrix $\mathcal{I}$, and the Lamé parameters μ and λ. The outward normal vectors to domains Ω_f and Ω_s are marked as $\mathbf{n}_f = (n_{f1}, n_{f2})^T$ and $\mathbf{n}_s = (n_{s1}, n_{s2})^T$.

The boundaries Γ_{0f} and Γ_{0s} are assumed to be rigid, whereas on the artificial boundaries Γ_{ef} and Γ_{es} we impose the conventional first order absorbing boundary conditions (see, e.g., [4]).

In addition to the system (1)-(8), we take into account the initial conditions $\mathbf{e}_0 = (\mathbf{e}_{0f}, \mathbf{e}_{0s})^T$ and $\mathbf{e}_1 = (\mathbf{e}_{1f}, \mathbf{e}_{1s})^T$, such that

$$p_f(\mathbf{x},0) = \mathbf{e}_{0f}, \quad \frac{\partial p_f}{\partial t}(\mathbf{x},0) = \mathbf{e}_{1f}, \quad \text{in } \Omega_f, \tag{9}$$

$$\mathbf{u}_s(\mathbf{x},0) = \mathbf{e}_{0s}, \quad \frac{\partial \mathbf{u}_s(\mathbf{x},0)}{\partial t} = \mathbf{e}_{1s}, \quad \text{in } \Omega_s. \tag{10}$$

Essentially, the solution procedure of the exact controllability problem is similar to those presented for the Helmholtz equation in [5] and for the Navier equation in [8]. After discretization, the exact controllability problem is reformulated as a least-squares optimization problem

$$\min \frac{1}{2}\left((\mathbf{U}(T) - \mathbf{e}_0)^T \begin{pmatrix} \mathcal{K}_s & 0 \\ 0 & \mathcal{K}_f, \end{pmatrix} (\mathbf{U}(T) - \mathbf{e}_0) + \left(\frac{\partial \mathbf{U}(T)}{\partial t} - \mathbf{e}_1 \right)^T \begin{pmatrix} \mathcal{M}_s & 0 \\ 0 & \mathcal{M}_f \end{pmatrix} \left(\frac{\partial \mathbf{U}(T)}{\partial t} - \mathbf{e}_1 \right) \right),$$

where $\mathbf{U}(T)$ is the global vector containing the values of the displacement $\mathbf{u}_s(\mathbf{x}, T)$ and the pressure $p_f(\mathbf{x}, T)$ at time T at the Gauss-Lobatto points of the spectral element mesh (see, e.g., [3, 6]). The stiffness matrices corresponding to solid and fluid parts are $\mathcal{K}_s$ and $\mathcal{K}_f$, while $\mathcal{M}_s$ and $\mathcal{M}_f$ are diagonal mass matrices representing elastic and acoustic phenomena, respectively.

The minimization problem is solved with a preconditioned conjugate gradient algorithm with the block-diagonal preconditioner diag $(\mathcal{K}_s, \mathcal{K}_f, \mathcal{M}_s, \mathcal{M}_f)$. For solving linear systems with stiffness matrices in preconditioning, we use the algebraic multigrid (AMG) method [7] (see also [1, 5]).

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