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# CORRELATION LENGTH CONTROLLABLE PRIORS IN STATISTICAL INVERSION

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**ABSTRACT**

In statistical inversion theory, the objective is to find an unknown function, such as magnetic field of the Earth. Typical linear inverse problems are formally described with the following operator equation,

$$y = Ax + \varepsilon, \tag{1}$$

where $x$ is the unknown. The operator $A$ is a known linear mapping. We also know the measurements $y$ and assume, that the measurements are spatially fixed, i.e. $y$ is a finite dimensional vector. Finally, we assume, that we know the statistical properties of the noise process $\varepsilon$.

If $A^{-1}$ is large or it does not even exist, the solution candidate given by $A^{-1}y$ is useless. Then we refer to the problem as unstable. In statistical inversion theory, one searches for a solution by adding some a priori information, in order to stabilize the solution. The solution is given by as an a posteriori distribution given as following distribution density.

$$D(x|y) \propto D_{\mathrm{pr}}(x)D(y|x) \tag{2}$$

By the proportionality sign, we mean that the a posteriori distribution is true, except for the normalization of the distribution.

In order to carry out the statistical inversion, one has to model the a priori distribution in some sensible way. This information can be given for example with a Gaussian process. Typically this information could be given for example as an Ornstein-Uhlenbeck prior, which is a zero-mean Gaussian process with covariance function

$$\mathrm{cov}(x_t, x_{t'}) = \alpha \exp\left(-\frac{1}{l}|t - t'|\right) \tag{3}$$

This prior is easy to use and discretize. It has a clear interpretation as the strength of the correlation $\alpha$ and as the length of correlation $l$. This prior has only one drawback. When discretized, the covariance matrix turns out to be a full matrix. In statistical inversion, one usually inverts the covariance matrix

when searching for example of the center-point estimate of the posterior distribution. Then it is clear, that high-dimensional inversion results cannot be computed.

Instead of Ornstein-Uhlenbeck prior, one searches normally a posteriori solution with smoothness priors, which are for example given as a difference prior

$$x_{i-1} - x_i \sim N(0, \sigma^2) \tag{4}$$

This prior is very easy to use. However, the interpretation of the prior is not clear, because the choice of the regularization parameter $\sigma^2$ can be very difficult. In (Lasanen et al. 2005), they showed, that the parameter can be given with the help of the discretization, given as follows.

$$x_i - x_{i-1} \sim N(0, \sigma^2 h) \tag{5}$$

This prior distribution has a clear interpretation in the sense, that the a posteriori distribution is essentially independent of the discretization. It also fixes the correlation length in some natural units, such as time. However, if we want to control the correlation length in a unified way, we have to use different kind of priors. Correlation length controllable priors fix this drawback. First order correlation length controllable prior can be given for example as following equation.

$$x_i \sim N(0, \alpha l/h) \tag{6}$$

$$x_i - x_{i-1} \sim N(0, \alpha h/l) \tag{7}$$

This prior has a finite correlation length $l$, it has a strength $\alpha$ and it is essentially independent of the discretization. Moreover, this a priori distribution corresponds to Ornstein-Uhlenbeck prior except for the boundary conditions. However, the correlation length controllable prior has a diagonal covariance matrix and therefore the computation time of the inverse estimates is much faster with correlation length controllable priors than with Ornstein-Uhlenbeck prior. For example the computation time for Fisher information matrix has a two decade difference. Therefore, we conclude, that we have found a computationally efficient exponential prior, whose regularization parameter has a clear and concise interpretation.

A generalization of the correlation length controllable priors is quite straightforward. Take a white noise process as the first counterpart and some degree difference prior as a second counterpart. Thereafter take account of the discretization as explained in (Lasanen et al. 2005). Then replace the rest of the regularization parameter by the strength of the regularization $\alpha$ and by the correlation length parameter $l$. This parameter $l$ is always inverse of the discretization parameter with the same power as the discretization parameter. For example second order correlation length controllable prior can be given as follows

$$x_i \sim N(0, \alpha l/h) \tag{8}$$

$$x_{i-1} - 2x_i + x_{i+1} \sim N(0, \alpha (h/l)^3) \tag{9}$$

The generalization to higher dimensions is done in a similar manner.

## REFERENCES


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