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Tackling geometric uncertainty with stochastic projection schemes

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ABSTRACT

Geometric uncertainties are commonly encountered in computational modeling of a wide variety of engineered and natural systems. A motivating example is statistical analysis of the biomechanical integrity of total hip replacements, where there is a pressing need to consider inter- and intra-patient variability in the femur geometry and density distribution. There is currently very little work in the literature on how to incorporate such models of geometric uncertainty in stochastic finite element methods.

Recently, Xiu and Tartakovsky [1] presented an elegant approach for solving partial differential equations (PDEs) on random domains. The central idea of the Xiu-Tartakovsky method is to create a spatial mapping that transforms the original PDE into a stochastic PDE defined on a *regular, deterministic* domain. Mohan et al. [2] later revisited this problem and developed a non-intrusive strategy that couples mesh morphing strategies with stochastic reduced basis methods.

The present work is concerned with the development of a general-purpose *intrusive* framework for incorporating probabilistic models of uncertain geometries in stochastic finite element analysis. The key objective is to enable the application of stochastic projection schemes such as those based on polynomial chaos (PC) expansions [3] and stochastic Krylov methods [4] to efficiently solve deterministic/stochastic PDEs on random domains. To illustrate the proposed approach, consider the linear deterministic operator problem $\mathcal{L}u(x;\omega) = f(x) \;\; \text{in} \;\; x \in \mathcal{D}(\omega)$, subject to the boundary condition $\mathcal{B}u(x;\omega) = g(x) \;\; \text{on} \;\; x \in \partial\mathcal{D}(\omega)$. Here, $\mathcal{L}$ and $\mathcal{B}$ are differential operators in space $\mathbb{R}^3$, $\mathcal{D}$ is a regular domain with $\partial\mathcal{D}(\omega)$ as its random boundary. $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, where Ω is the sample space, $\mathcal{F}$ is the $\sigma-$algebra associated with Ω, $\mathcal{P}$ is a probability measure, and $\omega \in \Omega$. $u(x;\omega)$ is the solution process whose statistics are of interest. Without any loss of generality, any random boundary can be represented as the zero level set of an appropriate function $r(x, \theta(\omega))$, i.e., $\partial\mathcal{D}(\omega) = \{x | r(x, \theta(\omega)) = 0, \omega \in \Omega\}$, where θ denotes a set of random variables with specified probability density function $p(\theta)$.

Semi-discretization procedures for stochastic PDEs have been extensively studied for problems where only the PDE coefficients are random [3]. However, when the domain of definition of the PDE is itself random, spatial discretization is no longer straightforward. The proposed approach to solve PDEs on

random domains involves transforming the original PDE into a set of two stochastic PDEs with one-way coupling. As discussed next, the first PDE expresses the dependance of the vertices of a spatial mesh on the uncertain parameters θ. The solution of this PDE is then used as an input for discretization of the original governing equations on a stochastic mesh with fixed connectivity but random vertices.

To enable semi-discretization of the governing equations, we model random geometries by the stochastic mesh representation $(K, V(\theta))$, where K is a simplicial complex representing the connectivity of the vertices/edges/faces and $V(\theta) = \{v_1(\theta), \dots, v_m(\theta)\}$ is a set of m vertex positions defining the shape of the mesh in $\mathbb{R}^3$. In other words, we model random geometries by a mesh with fixed connectivity but randomly located vertices. This representation is very flexible since it allows for a hybrid mix of various element-types such as tetrahedra and brick elements to accurately capture complex geometrical features given a sufficient number of elements. To proceed further, we need to establish a relationship between the given level set function $r(x, \theta)$ and the statistics of the vertices $V(\theta)$. We shall show that a PDE-based or a radial basis function formalism [5] can be readily adopted to achieve this goal.

To illustrate one possible approach motivated by elasticity theory, consider a mesh of the nominal domain represented by the pair $(K, V(\langle\theta\rangle))$ with m vertices. Our approach is based on the idea that geometric uncertainties can be modeled by a *prescribed* surface displacement field and the *nominal* domain can be thought of as an *virtual* elastic medium that deforms in response to it. This formalism enables the mesh vertices $V(\theta)$ to be readily computed in a PC basis via the solution of a deterministic PDE (with Navier elasticity operator) subject to random (displacement) boundary conditions specified in accordance with the level set function $r(x, \theta)$.

Given the mesh connectivity information K and a PC representation of the vertices $V(\theta)$, the original governing equations can be discretized in space to arrive at a system of random algebraic equations. In the case of finite element spatial discretization, this step will involve the computation of integrals over elements whose vertices are randomly distributed. The integrals that arise can be written in the general form $\int_{\mathcal{D}_e(\theta)} g(x;\omega)dx$, where $\mathcal{D}_e(\theta)$ denotes an element (e.g., tetrahedra) whose vertices are represented in a PC basis. We shall show how such integrals can be efficiently evaluated in a PC basis which in turn leads to PC expansions of the element-level coefficient matrices.

Assembly of the element matrices followed by application of boundary conditions result in a linear random algebraic system of equations of the form $A(\theta)x(\theta) = b(\theta)$, where $A(\theta)$ and $b(\theta)$ are represented in a PC basis. These equations can be readily solved using PC projection or stochastic Krylov methods for $x(\theta)$. We shall show that given the solution process $x(\theta)$ and the mesh vertices $V(\theta)$ in a PC basis, it becomes possible to efficiently compute various statistics of the solution process $u(x;\omega)$ in the post-processing stage.

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