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Efficient Finite Element Solvers for p-Laplacian Equation

*** E. Senger[1] and A. F. D. Loula[2]**

[1] Universidade Federal do Amapá
Rod. Juscelino Kubitschek, S/N, Macapá, AP - Brasil
erasmo@lncc.br

[2] Laboratório Nacional de Computação Científica
Av. Getúlio Vargas 333, Petrópolis, RJ - Brasil
aloc@lncc.br

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ABSTRACT

Let Ω be a bounded open subset of R^2 with a smooth boundary $\Gamma = \partial\Omega$. We will deal with finite element approximations of the following problem:

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = f \;\text{ in}\,\Omega \quad \text{and}\; u = 0 \;\text{ on }\; \partial\Gamma,$$

with $f \in L^2(\Omega)$, $1 < p < \infty$. The above nonlinear operator is known as the p-Laplacian and occurs in many mathematical models associated with glacial processes, image processing, nonlinear diffusions, filtration, creeping flows in solids and quasi-Newtonian flows in general. See [1-6] and references therein.

Constructing finite element approximation for this problem presents no particular difficulty. The main issue is solving the resulting nonlinear algebraic system for large values of the power p, that is $p \gg 2$, or for values of p close to one. Before introducing the finite element approximation we observe that the above p-Laplacian problem is equivalent to the minimization problem: Find $u \in W_0^{1,p}(\Omega)$ such that $J(u) \leq J(v)$ for all $v \in W_0^{1,p}(\Omega)$, with

$$J(v) = \frac{1}{p}\int_\Omega |\nabla v|^p d\Omega - \int_\Omega f v d\Omega\,,$$

or to the following weak formulation: Find $u \in W_0^{1,p}(\Omega)$ such that

$$(|\nabla u|^{p-2}\nabla u, \nabla v) = (f, v) \quad \forall v \in W_0^{1,p}(\Omega)$$

with $(\cdot,\cdot)$ denoting the duality product. Based on this weak form we construct finite element approximations on classical C^0 Lagrangian finite spaces $V_h \subset H_0^1(\Omega)$ leading to the finite dimension nonlinear problem: Find $u_h \in V_h$ such that

$$(|\nabla u_h|^{p-2}\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h\,.$$

To solve this nonlinear system efficiently, iterative solvers have been proposed as, for example, the hybrid conjugate gradient method with weighted preconditioner for the p-Laplacian [1], multigrid algorithms [2] or penalization techniques [3]. In this work we propose very simple iterative algorithms based on a quasi Newton method with relaxation which works very efficiently for very large values of p as well as p close to one. Alternatively, we develop an even simpler algorithm based on the following

constrained minimization problem: Find $\sigma \in L^q_f(\Omega) = \{\tau \in [L^q(\Omega)]^2\,,\ -\mathrm{div}\tau = f\ a.e.\ \text{in}\ \Omega\}$ such that $G(\sigma) \leq G(\tau)$ for all $\tau \in [L^q_f(\Omega)]^2$, $q = \frac{p}{p-1}$, with

$$G(\tau) = \frac{1}{p}\int_\Omega |\tau|^p d\Omega\,.$$

This constrained minimization problem is equivalent to the saddle-point problem of the Lagrangian

$$L(\tau, v) = G(\tau) + (\mathrm{div}\tau + f, v)$$

which gives rise to the following mixed formulation: Find $\sigma \in [L^q(\Omega)]^2$ and $u \in W^{1,p}_0(\Omega)$ such that

$$(|\sigma|^{q-2}\sigma, \tau) = (\nabla u, \tau) \quad \forall \tau \in [L^q(\Omega)]^2$$
$$(\sigma, \nabla v) = (f, v) \quad \forall v \in W^{1,q}_0(\Omega)\,.$$

Using the Helmholtz decomposition

$$\sigma = \nabla\phi + \mathbf{curl}\psi,$$

in which ϕ and ψ are scalar fields in $W^{1,q}_0(\Omega)$ and $W^{1,q}(\Omega)/\mathrm{R}$, respectively, and considering the first equation of the above mixed formulation in the inverse form

$$(\sigma, \tau) = (|\nabla u|^{p-2}, \tau) \quad \forall \tau \in [W^q(\Omega)]^2\,,$$

we obtain the following system of equations

$$(\nabla\phi, \nabla w) = (f, w) \quad \forall w \in W^{1,q}_0(\Omega)\,,$$
$$(\nabla u, \nabla v) = (|\nabla\phi + \mathbf{curl}\psi|^{q-2}(\nabla\phi + \mathbf{curl}\psi), \nabla v) \quad \forall v \in W^{1,q}_0(\Omega)\,,$$
$$(\mathbf{curl}\psi, \mathbf{curl}\eta) = (|\nabla u|^{p-2}\nabla u, \mathbf{curl}\eta) \quad \forall \eta \in W^{1,q}(\Omega)\,.$$

Note that ϕ is obtained solving a linear problem. The system in ψ and u is solved using the algorithm:

$$(\nabla u^{n+1}, \nabla v) = \zeta_n(\nabla u^n, \nabla v) + (1-\zeta_n)|\nabla\phi + \mathbf{curl}\psi^n|^{q-2}(\nabla\phi + \mathbf{curl}\psi^n), \nabla v) \quad \forall v \in W^{1,q}_0(\Omega)\,,$$
$$(\mathbf{curl}\psi^{n+1}, \mathbf{curl}\eta) = \rho_n(\mathbf{curl}\psi^n, \mathbf{curl}\eta) + \rho_n(|\nabla u^{n+1}|^{p-2}\nabla u^{n+1}, \mathbf{curl}\eta) \quad \forall \eta \in W^{1,q}(\Omega)$$

with $0 < \zeta_n < 1$ and $0 < \rho_n < 1$. With appropriate choice of the relaxation parameters ζ_n and ρ_n can handle efficiently problems with $p \gg 2$ or p close to one.

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