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A NEW SHALLOW WATER MODEL WITH EXPLICIT POLYNOMIAL DEPENDENCE ON DEPTH

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ABSTRACT

In this work, we study Euler equations in a domain with small depth. With this aim, we introduce a small adimensional parameter ε related to the depth, and the domain we consider can be defined by $\Omega^\varepsilon = \{(x^\varepsilon, y^\varepsilon, z^\varepsilon) \in \mathbb{R}^3 | (x^\varepsilon, y^\varepsilon) \in D, z^\varepsilon \in (H^\varepsilon(x^\varepsilon, y^\varepsilon), s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon))\}$ where D is the projection on the XY plane of Ω^ε, $z^\varepsilon = H^\varepsilon(x^\varepsilon, y^\varepsilon)$ is the equation of the bottom of the domain (supposed known), $z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)$ is the equation of the free surface (unknown) and we define the water depth as $h^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) - H^\varepsilon(x^\varepsilon, y^\varepsilon)$. The adimensional parameter ε allows us to state explicitly that the domain has small depth by supposing that $H^\varepsilon = \varepsilon H$, $s^\varepsilon = \varepsilon s$, $h^\varepsilon = \varepsilon h$ and $x^\varepsilon = x$, $y^\varepsilon = y$, $t^\varepsilon = t$ (that is, x^ε, y^ε and t^ε are independent of ε). In this way, ε can be thought as the quotient between characteristic depth and diameter of the domain.

Let us consider that flow obeys Euler equations in Ω^ε and that the external forces acting on the fluid are those due to gravity and the Coriolis acceleration. The fluid is supposed to be incompressible and the pressure is the atmospheric at the surface.

Usually, when used asymptotic analysis to analyze fluids, most of authors work in the original domain (see, for example [3]), or they suppose that the surface is flat (see, for example, [1]). We, however, shall use the asymptotic technique in the same way as in [2] and related works, that is, we make a change of variable to a reference domain independent of the parameter ε and time.

Let $\Omega = D \times (0,1)$ be the reference domain and let us define the following change of variable, from Ω to Ω^ε: $t^\varepsilon = t, x^\varepsilon = x, y^\varepsilon = y, z^\varepsilon = \varepsilon[H(x,y) + zh(t,x,y)]$. Given any function F^ε defined on $[0,T] \times \overline{\Omega^\varepsilon}$, we can define other function $F(\varepsilon)$ on $[0,T] \times \bar{\Omega}$ using the change of variable: $F(\varepsilon)(t,x,y,z) = F^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon)$. Now our problem can be written in the reference domain Ω with explicit dependence on ε.

In order to apply the formal asymptotic method, we assume that the solution to the problem in the reference domain allows an expansion in powers of ε. We replace this expansion in the equations obtained, after the change of variable, in Ω and we identify the terms multiplied by the same power of ε. In this way we arrive at a series of equations that will allow us to determine each term of the expansion. If

we neglect the terms of bigger order in the equations we obtain, we attain the following shallow water model expressed in terms of the depth averaged velocity, whose order of precision (at least formally) is $O(\varepsilon^2)$:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}(h^\varepsilon \bar{\vec{\mathbf{u}}}^\varepsilon) = 0, \quad \frac{\partial \bar{\vec{\mathbf{u}}}^\varepsilon}{\partial t^\varepsilon} + \nabla \bar{\vec{\mathbf{u}}}^\varepsilon \cdot \bar{\vec{\mathbf{u}}}^\varepsilon + g\nabla h^\varepsilon = -\frac{1}{\rho_0}\nabla p_s^\varepsilon - g\nabla H^\varepsilon + 2\phi \vec{\mathbf{F}}_C^\varepsilon \tag{1}$$

$$\frac{\partial \vec{\gamma}^{j,\varepsilon}}{\partial t^\varepsilon} + \nabla \vec{\gamma}^{j,\varepsilon} \cdot \bar{\vec{\mathbf{u}}}^\varepsilon - (\nabla \bar{\vec{\mathbf{u}}}^\varepsilon)^T \cdot \vec{\gamma}^{j,\varepsilon} = 2\phi \vec{\mathbf{F}}_V^{j,\varepsilon} \quad (j = 0, 1, \ldots, N) \tag{2}$$

$$u^\varepsilon = \bar{u}^\varepsilon + \sum_{j=0}^{N} \left[\gamma_2^{j,\varepsilon} \left(\frac{(z^\varepsilon - H^\varepsilon)^{j+1}}{(j+1)(h^\varepsilon)^j} - \frac{h^\varepsilon}{(j+1)(j+2)} \right) \right] \tag{3}$$

$$v^\varepsilon = \bar{v}^\varepsilon - \sum_{j=0}^{N} \left[\gamma_1^{j,\varepsilon} \left(\frac{(z^\varepsilon - H^\varepsilon)^{j+1}}{(j+1)(h^\varepsilon)^j} - \frac{h^\varepsilon}{(j+1)(j+2)} \right) \right] \tag{4}$$

$$\vec{\gamma}^\varepsilon = \sum_{j=0}^{N} \left(\frac{z^\varepsilon - H^\varepsilon}{h^\varepsilon} \right)^j \vec{\gamma}^{j,\varepsilon}, \qquad p^\varepsilon = p_s^\varepsilon + \rho_0 (s^\varepsilon - z^\varepsilon)[g - 2\phi(\cos\phi^\varepsilon)\bar{u}^\varepsilon] \tag{5}$$

where we denote by $\vec{\mathbf{u}}^\varepsilon = (u^\varepsilon, v^\varepsilon)$ the horizontal velocity, $\bar{\vec{\mathbf{u}}}^\varepsilon = (\bar{u}^\varepsilon, \bar{v}^\varepsilon)$ the averaged horizontal velocity, $\vec{\gamma}^\varepsilon = (\gamma_1^\varepsilon, \gamma_2^\varepsilon)$ the two first components of the vorticity ($\vec{\gamma}^{j,\varepsilon} = (\gamma_1^{j,\varepsilon}, \gamma_2^{j,\varepsilon})$), p^ε the pressure (p_s^ε the pressure at the surface), ϕ the angular velocity of the Earth, φ^ε the North latitude, g the gravity acceleration, ρ_0 the density,

$$\vec{\mathbf{F}}_C^\varepsilon = \begin{pmatrix} (\sin\varphi^\varepsilon)\bar{v}^\varepsilon + (\cos\varphi^\varepsilon)\left(\frac{\partial (h^\varepsilon \bar{u}^\varepsilon)}{\partial x^\varepsilon} + \frac{h^\varepsilon}{2}\frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} - \bar{v}^\varepsilon \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \\ -(\sin\varphi^\varepsilon)\bar{u}^\varepsilon + \frac{h^\varepsilon}{2}\frac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] + \frac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] \end{pmatrix} \tag{6}$$

$$\vec{\mathbf{F}}_V^{0,\varepsilon} = \begin{pmatrix} (\sin\varphi^\varepsilon)\gamma_2^{0,\varepsilon} + \frac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon] \\ -(\sin\varphi^\varepsilon)\gamma_1^{0,\varepsilon} + (\cos\varphi^\varepsilon)\frac{\partial \bar{v}^\varepsilon}{\partial y^\varepsilon} \end{pmatrix}, \ \vec{\mathbf{F}}_V^{j,\varepsilon} = (\sin\varphi^\varepsilon) \begin{pmatrix} \gamma_2^{j,\varepsilon} \\ \gamma_1^{j,\varepsilon} \end{pmatrix} \ (j = 1, \ldots, N) \tag{7}$$

and where $\operatorname{curl}\vec{\alpha}^\varepsilon = \frac{\partial \alpha_2^\varepsilon}{\partial x^\varepsilon} - \frac{\partial \alpha_1^\varepsilon}{\partial y^\varepsilon}$, $\operatorname{curl}\alpha^\varepsilon = \left(\frac{\partial \alpha^\varepsilon}{\partial y^\varepsilon}, -\frac{\partial \alpha^\varepsilon}{\partial x^\varepsilon} \right)$.

The formal asymptotic analysis allows us to obtain a shallow water model (1)-(5) that generalizes the classic shallow water model (1), providing a horizontal velocity with explicit dependence on z^ε. Numerical experiments confirm that our model (1)-(5) gives the same precision for all z^ε (if N is large enough) than the classic model for the averaged velocity, so we can consider our model as an improvement of the classical model.

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