ACCURATE SOLUTION OF GLOBAL FIELD INTERPOLATIONS FOR PARTICLE SIMULATIONS

L. A. Barba[1] and Louis F. Rossi[2]

[1] Dept. of Mathematics, University of Bristol
University Walk, Bristol, BS8 1TW, U.K.
l.a.barba@bristol.ac.uk

[2] Dept. of Mathematical Sciences, University of
Delaware, Newark, DL 19716, U.S.A.
rossi@math.udel.edu

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ABSTRACT

The general problem of finding a set of basis functions which accurately represents a given function is encountered often in particle simulations. It is also encountered of course in the basic interpolation problem, where the value of the field or function is known at a set of points, and basis functions centered on those points have to be found such that their superposition can represent the original unknown function accurately. When the basis functions used are multi-quadrics, thin-plate splines or Gaussians, which all have global support, the interpolation can be very accurate, but it is difficult to obtain due to the dense systems which need to be solved. In view of this problem, many researchers in the area of radial basis function interpolation have opted for bases of compact support, which result in an easier to solve sparse system, with a compromise in the interpolation accuracy which can be obtained.

In particle methods such as the vortex method for the simulation of incompressible fluid flow, one first encounters such an interpolation problem at the initialization stage of an initial value problem. When an initial vorticity distribution needs to be discretized using vortex particles, one chooses the locations or centres of the particles, and the size or radial spread of the bases (in the vortex method, this would be the smallest scales that can be resolved). It is common to use a Gaussian basis function in this application. What remains is to obtain the appropriate weights of each particle, such that their superposition represents well the initial vorticity.

For a straightforward but low accuracy solution of this problem, it has been standard for many years to initialize the vorticity on a regular lattice with separation h, and give the particles a weight γ by the assignment $\gamma_i = \omega_i h^2$ (in 2D; h^3 needs to be used instead in 3D). Here, ω_i represents the vorticity value (the function being interpolated) at each particle location. This method is similar to a midpoint approximation and does not require the solution of a linear system. It can be used where it is expected that spatial or temporal errors will be larger than the interpolation error and thus will dominate the simulation. Once the weights γ_i are found, the vorticity field becomes represented by: $\omega(x) = \sum_i \gamma_i \zeta_\sigma(x - x_i)$, where ζ represents the basis function and σ its radial size.

There is a second instance when field interpolation is needed in a vortex particle simulation, and it is when one wishes to replace a set of particles after they have been deformed by the flow strain. Lagrangian effects in this method cause particles to become clustered in certain directions and open gaps

in other areas, such that the particles no longer overlap and their superposition no longer represents a continuous field accurately. Many researchers have addressed this problem of replacing a disordered configuration of particles with a uniform one using some form of short-range interpolation. One popular interpolation method is the use of tensor products of splines, such as those used in the smoothed particle hydrodynamics method, applied to the circulation (weights) of the particles. This approach is widely used today by vortex method workers to perform long-time simulations.

If one seeks a high accuracy interpolation, the methods described above for initialization and spatial adaptation can be improved. In this vein, the use of radial basis function interpolation was proposed and demonstrated in [1]. One reason to want a more accurate field interpolation method is to take advantage of the high spatial order techniques using dynamically adaptive elliptical basis functions [2].

Most global field interpolation methods rely on the solution of a linear system $A\gamma = \omega$, where γ is a vector holding the weights, and ω is the vector of vorticty values. The matrix A maps the circulations to vorticity as the linear combination of basis functions which represents the vortex method discretization. The matrix is ill-conditioned, because basis functions overlap to provide good spatial accuracy. One can use some iterative solver, such as GMREs, to obtain the particle weights, but some kind of preconditioner will be essential for effective solution of the system.

We investigate some new methods for performing accurate field interpolations in the applications described above. For performing a preconditioned iterative solution of the dense linear system, we developed a preconditioner based on using the inverse of a sparse approximation to the matrix A. The sparse matrix is obtained by truncating the tails of the Gaussian basis functions, and the cutoff length for getting the sparse approximate matrix determines the convergence rate and the accuracy of the result. Using this preconditioner, we can solve the preconditioned system to machine precision in two or three GMRES iterations. Second, we borrow techniques from image processing to improve the naive approximation $\gamma_i = \omega_i h^2$. It can be shown that this approximation amounts to a Gaussian blurring of the true vorticity field via Newtonian diffusion. To improve the accuracy of the representation using the sum of Gaussians, we can solve the reverse heat equation for time $\Delta t = h$. This system can be solved using direct, explicit difference schemes to achieve a high accuracy approximation with small computational effort. So far, the accuracy improvement is not as considerable as when solving the full linear system, but of course the computational effort is much smaller.

We demonstrate the efficacy of the techniques we developed by performing computations of inviscid and viscous axisymmetrization of elliptical vortices. At high Reynolds numbers, these flows result in highly elliptical configurations which can be unstable and eject long, thin filaments of vorticity. To resolve these features with modest computational resources, high order methods are required and field interpolation is essential for maintaining accuracy over moderate flow times. We use this challenging set of problems to compare various field interpolation techniques.

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