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Remarks on the links between low order DG methods and some finite difference schemes for the Stokes problem

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ABSTRACT

Kanschat (2007) has shown that with a proper quadrature, the LDG method of Cockburn et al. (2005) can be made algebraically equivalent to the MAC scheme. The LDG method is based on the H^{div}-conforming Raviart-Thomas RT_0 element. In the present study we show that the Nédélec space of first kind, within the LDG setting, can yield a scheme which is also algebraically equivalent to the MAC method. In addition, we show that a piecewise constant approximation for the velocity can yield two other popular finite difference schemes proposed several decades ago.

The approximations to the Stokes equations employed in the present study are based on the piecewise solenoidal discrete spaces introduced in Baker et al. (1990) (see also Karakashian and Jureidini, 1998). We demonstrate that the piecewise constant approximation for the velocity on a uniform rectangular grid and the C^0 piecewise linear approximation for the pressure on a uniform triangular grid produced by subdividing the rectangles with their diagonals, yield a linear system equivalent to the system resulting from a finite difference scheme proposed by Kuznetsov (1968) and Fortin et al. (1971). This finite difference scheme is $O(h^2)$ consistent approximation of both the Laplace operator and the divergence/gradient operators. The latter scheme is also equivalent to the scheme proposed by Bell et al. (1989) up to the treatment of the boundary conditions. The higher order locally solenoidal approximations are related to the Nédélec spaces which are H^{curl}-conforming and are used mostly for the discretization of the Maxwell equations. The first order approximation for the velocity and the C^0 piecewise bi-linear approximation for the pressure on a uniform rectangular grid yield a scheme which can be made algebraically equivalent to the MAC scheme. It is remarkable, that the discretization of the Laplace operator is $O(h^4)$ consistent. It is the discretization of the gradient/divergence that reduces the consistency of the scheme to second order. Both schemes are $O(h^2)$ accurate (in a discrete maximum norm) and so the solution can be re-interpolated to obtain a fully second order (in an L^2 norm) approximation. Standard approximation theory with such interpolation spaces cannot yield such a high convergence rate and therefore this hints at a possible superconvergence on uniform grids. The first scheme does not satisfy the $inf-sup$ stability condition. However, the pressure space contains only one spurious checker-board mode which can be easily eliminated.

These observations allow to use the LDG setting to generalize the corresponding finite difference methods to unstructured grids or grids with non-matching nodes. The generalization of the first scheme to 3D

is straightforward. The generalization of the second scheme to 3D involves degrees of freedom for the velocity on the edges rather than faces of the elements and follows the same idea as the generalization of the Nédélec elements.

In addition, we also propose a procedure for constructing of a divergence free basis in the case of Nédélec approximation. The numerical tests show that, as it can be expected, the resulting linear system from the Stokes problem has a biharmonic conditioning. However, in case of large Reynolds numbers, this system may be preferable to solve and therefore, it can be a viable alternative to Uzawa iterations or projection. This basis can also be used for the resolution of the Maxwell equations in the limit of vanishing wave numbers. The same procedure can be employed for the construction of a divergence free basis in the case of Raviart-Thomas elements.

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